Marius Ionescu


April 18, 2011
An iterated function system (i.f.s.) is a collection \( \{ F_1, \ldots, F_N \} \) of contractions on \( \mathbb{R}^d \). For such an i.f.s. there exists a unique self-similar set \( K \) satisfying

\[
K = F_1(K) \cup \cdots \cup F_N(K).
\]

Let \( \mu \) be a self-similar measure on \( K \)

\[
\mu(A) = \frac{1}{N} \sum_{i=1}^{N} \mu(F_i^{-1}(A)).
\]
Definition

- An iterated function system (i.f.s.) is a collection \( \{F_1, \ldots, F_N\} \) of contractions on \( \mathbb{R}^d \).
Fractals

Definition

- An iterated function system (i.f.s.) is a collection \{F_1, \ldots, F_N\} of contractions on \(\mathbb{R}^d\).
- For such an i.f.s. there exists a unique self-similar set \(K\) satisfying

\[ K = F_1(K) \cup \cdots \cup F_N(K). \]
Definition

- An iterated function system (i.f.s.) is a collection \( \{F_1, \ldots, F_N\} \) of contractions on \( \mathbb{R}^d \).
- For such an i.f.s. there exists a unique self-similar set \( K \) satisfying
  \[
  K = F_1(K) \cup \cdots \cup F_N(K).
  \]
- Let \( \mu \) be a self-similar measure on \( K \)
  \[
  \mu(A) = \frac{1}{N} \sum_{i=1}^{N} \mu(F_i^{-1}(A)).
  \]
Definition

PCF Fractals

Definition

A PCF fractal is a post-critically finite self-similar set if there is a subset $V_0 \subseteq \{x_1, \ldots, x_N\}$ satisfying

$$F_\omega(K) \cap F_\omega'(K) \subseteq F_\omega(V_0) \cap F_\omega'(V_0).$$

$V_0$ is the boundary of the fractal.

$K_\omega = F_\omega(K)$ is called an $m$-cell if the length of $\omega$ is $m$.

$V_m = \bigcup \omega F_\omega(V_0)$ are the vertices and the $m$-graph approximation of $K$.

The set $V^* = \bigcup_m V_m$ is dense in $K$.

Marius Ionescu

The Resolvent Kernel For PCF Self-Similar Fractals
Definition

- $K$ is a post-critically finite (PCF) self-similar set if there is a subset $V_0 \subseteq \{x_1, \ldots, x_N\}$ satisfying

$$F_\omega(K) \cap F_\omega'(K) \subseteq F_\omega(V_0) \cap F_\omega'(V_0).$$
Definition

- $K$ is a post-critically finite (PCF) self-similar set if there is a subset $V_0 \subseteq \{x_1, \ldots, x_N\}$ satisfying

$$F_\omega(K) \cap F_\omega'(K) \subseteq F_\omega(V_0) \cap F_\omega'(V_0).$$

- $V_0$ is the boundary of the fractal.
Definition

- $K$ is a post-critically finite (PCF) self-similar set if there is a subset $V_0 \subseteq \{x_1, \ldots, x_N\}$ satisfying

$$F_\omega(K) \cap F_{\omega'}(K) \subseteq F_\omega(V_0) \cap F_{\omega'}(V_0).$$

- $V_0$ is the boundary of the fractal.
- $K_\omega = F_\omega(K)$ is called an $m$-cell if the length of $\omega$ is $m$. 

Marius Ionescu
The Resolvent Kernel For PCF Self-Similar Fractals
**Definition**

- $K$ is a **post-critically finite (PCF) self-similar** set if there is a subset $V_0 \subseteq \{x_1, \ldots, x_N\}$ satisfying
  \[
  F_\omega(K) \bigcap F_{\omega'}(K) \subseteq F_\omega(V_0) \bigcap F_{\omega'}(V_0).
  \]

- $V_0$ is the boundary of the fractal.
- $K_\omega = F_\omega(K)$ is called an $m$-cell if the length of $\omega$ is $m$.
- $V_m = \bigcup_\omega F_\omega(V_0)$ are the vertices and the $m$-graph approximation of $K$.
**Definition**

- $K$ is a *post-critically finite (PCF) self-similar* set if there is a subset $V_0 \subseteq \{x_1, \ldots, x_N\}$ satisfying

$$F_\omega(K) \cap F_{\omega'}(K) \subseteq F_\omega(V_0) \cap F_{\omega'}(V_0).$$

- $V_0$ is the boundary of the fractal.
- $K_\omega = F_\omega(K)$ is called an *$m$-cell* if the length of $\omega$ is $m$.
- $V_m = \bigcup_\omega F_\omega(V_0)$ are the vertices and the *$m$-graph approximation* of $K$.
- The set $V_* = \bigcup_m V_m$ is *dense* in $K$. 

---

**PCF Fractals**

- Marius Ionescu
- The Resolvent Kernel For PCF Self-Similar Fractals
We assume the existence of a self-similar energy form 

\[ E(u) = \sum_{i=1}^{r-1} r_i E(u \circ F_i) \].

This form is obtained as the limit of the normalized energy at level \( m \):

\[ E_m(u) = \sum_{x \sim y} c_{xy} (u(x) - u(y))^2. \]
We assume the existence of a *self-similar* energy form

\[ \mathcal{E}(u) = \sum_{i=1}^{N} r_i^{-1} \mathcal{E}(u \circ F_i). \]
We assume the existence of a self-similar energy form

\[ \mathcal{E}(u) = \sum_{i=1}^{N} r_i^{-1} \mathcal{E}(u \circ F_i). \]

This form is obtained as the limit of the normalized energy at level \( m \):

\[ \mathcal{E}_m(u) = \sum_{x \sim y} c_{xy} (u(x) - u(y))^2. \]
The Laplacian is defined weakly as
\[ E(u, v) = -\int_v \Delta u \, d\mu. \]
The pointwise formula
\[ \Delta u(x) = \lim_{m \to \infty} c_m(x) \Delta_m(x), \]
where \( \Delta_m \) is the Laplacian of the \( m \)-level graph.
The Laplacian is defined weakly

\[ \mathcal{E}(u, v) = -\int v \Delta u d\mu. \]
The Laplacian is defined weakly

\[ \mathcal{E}(u, v) = -\int v \Delta u \, d\mu. \]

The pointwise formula

\[ \Delta u(x) = \lim_{m \to \infty} c_m(x) \Delta_m(x), \]

where \( \Delta_m \) is the Laplacian of the \( m \)-level graph.
The normal derivative of a function at a boundary point $q$ is defined as:

$$
\partial_n u(q) = \lim_{{m \to \infty}} \frac{1}{r_i^m} \sum_{{y \sim q}} (u(q) - u(y)).
$$
Theorem

Assume that $\lambda$ is not a Dirichlet eigenvalue of $\Delta$, and neither is $\frac{1}{N^m} r_\omega \lambda$, for any finite word $\omega$. For the Laplacian on $K$ with Dirichlet boundary conditions the solution of the equation

$$(\lambda - \Delta) u = f$$

is given by integration with respect to a resolvent kernel $R^{(\lambda)}(x, y)$:

$$u(y) = \int R^{(\lambda)}(x, y) f(y) d\mu(y).$$
Fact

The resolvent kernel is built as follows:

First we solve the resolvent equation at level 1:

\[(\lambda - \Delta) \psi(\lambda) p = 0,\]
on each \(K_j = F_j(K)\), for \(p \in V_1 \setminus V_0\) and \(q \in V_1\).

Then we build a matrix \(B(\lambda)\) with the following entries:

\[B(\lambda) p q = \sum \partial_n \psi(\lambda) p(q).\]
The resolvent kernel is built as follows:

First we solve the resolvent equation at level 1:

\[
\begin{cases}
(\lambda - \Delta)\psi_p^{(\lambda)} = 0, & \text{on each } K_j = F_j(K), \\
\psi_p^{(\lambda)}(q) = \delta_{pq}, & \text{for } p \in V_1 \setminus V_0 \text{ and } q \in V_1.
\end{cases}
\]
The resolvent kernel is built as follows:

- First we solve the resolvent equation at level 1:
  \[
  \begin{align*}
  (\lambda - \Delta)\psi_{p}^{(\lambda)} &= 0, \quad \text{on each } K_j = F_j(K), \\
  \psi_{p}^{(\lambda)}(q) &= \delta_{pq}, \quad \text{for } p \in V_1 \setminus V_0 \text{ and } q \in V_1.
  \end{align*}
  \]

- Then we build a matrix \( B^{(\lambda)} \) with the following entries
  \[
  B_{pq}^{(\lambda)} = \sum \partial_n \psi_{p}^{(\lambda)}(q).
  \]
Fact

If $\lambda$ is not a Dirichlet eigenvalue, then $B(\lambda)$ is an invertible matrix.

Let $G(\lambda)$ be the inverse on $B(\lambda)$.

Define the map $\Psi(\lambda)(x, y) = \sum_{p, q \in V_1 \setminus V_0} G(\lambda)_{pq} \psi(\lambda)_p(x) \psi(\lambda)_q(y)$.

Finally, the resolvent kernel is given by $R(\lambda)(x, y) = \sum_{\omega \in \Omega} w_{\omega} \Psi(\lambda)(F^{-1}w_{\omega}x, F^{-1}w_{\omega}y)$. 

Marius Ionescu

The Resolvent Kernel For PCF Self-Similar Fractals
Fact

- If \( \lambda \) is not a Dirichlet eigenvalue, then \( B^{(\lambda)} \) is an invertible matrix.

Finally, the resolvent kernel is given by

\[
R^{(\lambda)}(x, y) = \sum_{\omega \in r} \sum_{w} \Psi^{(1)}(N_{m \omega}^{(r \omega)}(\lambda)) \left(F^{-1}_{w}(x), F^{-1}_{w}(y)\right).
\]
Fact

- If $\lambda$ is not a Dirichlet eigenvalue, then $B^{(\lambda)}$ is an invertible matrix.
- Let $G^{(\lambda)}$ be the inverse on $B^{(\lambda)}$. 
Fact

- If $\lambda$ is not a Dirichlet eigenvalue, then $B^{(\lambda)}$ is an invertible matrix.
- Let $G^{(\lambda)}$ be the inverse on $B^{(\lambda)}$.
- Define the map

$$\Psi^{(\lambda)}(x, y) = \sum_{p, q \in V_1 \setminus V_0} G^{(\lambda)}_{pq} \psi_p^{(\lambda)}(x) \psi_q^{(\lambda)}(y).$$
Fact

- If $\lambda$ is not a Dirichlet eigenvalue, then $B(\lambda)$ is an invertible matrix.
- Let $G(\lambda)$ be the inverse on $B(\lambda)$.
- Define the map

$$\psi(\lambda)(x, y) = \sum_{p,q \in V_1 \setminus V_0} G_{pq}^{(\lambda)} \psi_p^{(\lambda)}(x) \psi_q^{(\lambda)}(y).$$

- Finally, the resolvent kernel is given by

$$R^{(\lambda)}(x, y) = \sum_{\omega} r_w \psi^{(\frac{1}{N^m} r_w \lambda)}(F_w^{-1} x, F_w^{-1} y).$$
Example: Unit interval

Example

For the unit interval we have that

\[
\psi^{(\lambda)}(x) = \frac{1}{\sinh \frac{\sqrt{\lambda}}{2}} \begin{cases} 
\sinh \sqrt{\lambda} x & x \leq \frac{1}{2} \\
\sinh \sqrt{\lambda} (1 - x) & x \geq \frac{1}{2}
\end{cases},
\]

\[
\Psi^{(\lambda)}(x, y) = \sinh \sqrt{\lambda} 2 \sqrt{\lambda} \cosh \sqrt{\lambda} 2 \psi^{(\lambda)}(x) \psi^{(\lambda)}(y),
\]

\[
R^{(\lambda)}(x, y) = \infty \sum_{m=0}^{\infty} \sum_{|\omega|=m} \frac{1}{2} \Psi^{(\lambda/4)}(F^{-1}\omega x, F^{-1}\omega y).
\]
Example: Unit interval

For the unit interval we have that

\[
\psi^{(\lambda)}(x) = \frac{1}{\sinh \frac{\sqrt{\lambda}}{2}} \begin{cases} 
\sinh \sqrt{\lambda} x & x \leq \frac{1}{2} \\
\sinh \sqrt{\lambda} (1 - x) & x \geq \frac{1}{2}
\end{cases},
\]

\[
\Psi^{(\lambda)}(x, y) = \frac{\sinh \frac{\sqrt{\lambda}}{2}}{2\sqrt{\lambda} \cosh \frac{\sqrt{\lambda}}{2}} \psi^{(\lambda)}(x) \psi^{(\lambda)}(y),
\]
Example: Unit interval

For the unit interval we have that

\[ \psi^{(\lambda)}(x) = \frac{1}{\sinh \frac{\sqrt{\lambda}}{2}} \begin{cases} \sinh \sqrt{\lambda} x & x \leq \frac{1}{2} \\ \sinh \sqrt{\lambda} (1 - x) & x \geq \frac{1}{2} \end{cases}, \]

\[ \Psi^{(\lambda)}(x, y) = \frac{\sinh \frac{\sqrt{\lambda}}{2}}{2 \sqrt{\lambda} \cosh \frac{\sqrt{\lambda}}{2}} \psi^{(\lambda)}(x) \psi^{(\lambda)}(y), \]

and

\[ R^{(\lambda)}(x, y) = \sum_{m=0}^{\infty} \sum_{|\omega|=m} \frac{1}{2^m} \psi^{(\lambda/4^m)}(F^{-1}_\omega x, F^{-1}_\omega y). \]
Unit interval: picture of $\psi^{(1)}(x, y)$

(a) 

(b) 

(c)
Unit interval: picture of $R^{(1)}(x, y)$
For the Sierpinski gasket the matrix $G^{(\lambda)}$ is given by

$$G^{(\lambda)} = \frac{3}{5(5 - \lambda_0)(2 - \lambda_0)\tau(\lambda)} \begin{bmatrix}
3 - \lambda_0 & 1 & 1 \\
1 & 3 - \lambda_0 & 1 \\
1 & 1 & 3 - \lambda_0
\end{bmatrix},$$

where

$$\tau(\lambda) = \frac{4\lambda}{3\lambda_0(2 - \lambda_1)} \prod_{j=2}^{\infty} \left(1 - \frac{\lambda_j}{3}\right).$$