The Resolvent Kernel For PCF Self-Similar Fractals

Marius Ionescu

joint with Erin P. J. Pearse, Luke G. Rogers, Huo-Jun Ruan, Robert S. Strichartz

April 18, 2011

Definition

Fractals

Definition

- An iterated function system (i.f.s.) is a collection $\{F_1, \ldots, F_N\}$ of contractions on $\mathbb{R}^d$.

Definition

- An iterated function system (i.f.s.) is a collection $\{F_1, \ldots, F_N\}$ of contractions on $\mathbb{R}^d$.
- For such an i.f.s. there exists a unique self-similar set K satisfying
$$K = F_1(K) \bigcup \cdots \bigcup F_N(K).$$

Definition

- An iterated function system (i.f.s.) is a collection $\{F_1, \ldots, F_N\}$ of contractions on $\mathbb{R}^d$.
- For such an i.f.s. there exists a unique self-similar set K satisfying
$$K = F_1(K) \bigcup \cdots \bigcup F_N(K).$$
- Let μ be a self-similar *measure* on K
$$\mu(A) = \frac{1}{N} \sum_{i=1}^{N} \mu(F_i^{-1}(A)).$$

PCF Fractals

Definition

Definition

- K is a *post-critically finite (PCF) self-similar* set if there is a subset $V_0 \subseteq \{x_1, \dots, x_N\}$ satisfying

$$F_\omega(K) \bigcap F_{\omega'}(K) \subseteq F_\omega(V_0) \bigcap F_{\omega'}(V_0).$$

PCF Fractals

Definition

- K is a *post-critically finite (PCF) self-similar* set if there is a subset $V_0 \subseteq \{x_1, \ldots, x_N\}$ satisfying

$$F_\omega(K) \bigcap F_{\omega'}(K) \subseteq F_\omega(V_0) \bigcap F_{\omega'}(V_0).$$

- V_0 is the boundary of the fractal.

PCF Fractals

Definition

- K is a *post-critically finite (PCF) self-similar* set if there is a subset $V_0 \subseteq \{x_1, \ldots, x_N\}$ satisfying

$$F_\omega(K) \bigcap F_{\omega'}(K) \subseteq F_\omega(V_0) \bigcap F_{\omega'}(V_0).$$

- V_0 is the boundary of the fractal.
- $K_\omega = F_\omega(K)$ is called an m-cell if the length of ω is m.

PCF Fractals

Definition

- K is a *post-critically finite (PCF) self-similar* set if there is a subset $V_0 \subseteq \{x_1, \dots, x_N\}$ satisfying

$$F_\omega(K) \bigcap F_{\omega'}(K) \subseteq F_\omega(V_0) \bigcap F_{\omega'}(V_0).$$

- V_0 is the boundary of the fractal.
- $K_\omega = F_\omega(K)$ is called an m-cell if the length of ω is m.
- $V_m = \bigcup_\omega F_\omega(V_0)$ are the vertices and the m-graph approximation of K

PCF Fractals

Definition

- K is a *post-critically finite (PCF) self-similar* set if there is a subset $V_0 \subseteq \{x_1, \ldots, x_N\}$ satisfying

$$F_\omega(K) \bigcap F_{\omega'}(K) \subseteq F_\omega(V_0) \bigcap F_{\omega'}(V_0).$$

- V_0 is the boundary of the fractal.
- $K_\omega = F_\omega(K)$ is called an m-cell if the length of ω is m.
- $V_m = \bigcup_\omega F_\omega(V_0)$ are the vertices and the m-graph approximation of K
- The set $V_* = \bigcup_m V_m$ is *dense* in K.

Energy ...

Definition

Energy ...

Definition

- We assume the existence of a *self-similar* energy form

$$\mathcal{E}(u) = \sum_{i=1}^{N} r_i^{-1}\mathcal{E}(u \circ F_i).$$

Definition

- We assume the existence of a *self-similar* energy form

$$\mathcal{E}(u) = \sum_{i=1}^{N} r_i^{-1} \mathcal{E}(u \circ F_i).$$

- This form is obtained as the limit of the normalized energy at level m:

$$\mathcal{E}_m(u) = \sum_{x \tilde{_m} y} c_{xy} (u(x) - u(y))^2.$$

... and Laplacian

Definition

Definition

- The *Laplacian* is defined *weakly*

$$\mathcal{E}(u,v) = -\int v\Delta u d\mu.$$

Definition

- The *Laplacian* is defined *weakly*

$$\mathcal{E}(u,v) = -\int v\Delta u d\mu.$$

- The *pointwise* formula

$$\Delta u(x) = \lim_{m\to\infty} c_m(x)\Delta_m(x),$$

where Δ_m is the Laplacian of the m-level graph.

Normal Derivative

Definition

The *normal derivative* of a function at a boundary point q is defined

$$\partial_n u(q) = \lim_{m\to\infty} \frac{1}{r_i^m} \sum_{y \tilde{m} q} (u(q) - u(y)).$$

Main Result

Theorem

Assume that λ is not a Dirichlet eigenvalue of Δ, and neither is $\frac{1}{N^m} r_\omega \lambda$, for any finite word ω. For the Laplacian on K with Dirichlet boundary conditions the solution of the equation

$$(\lambda - \Delta)u = f$$

is given by integration with respect to a resolvent kernel $R^{(\lambda)}(x, y)$:

$$u(y) = \int R^{(\lambda)}(x, y) f(y) d\mu(y).$$

Formula for the resolvent kernel

Fact

The resolvent kernel is built as follows:

Formula for the resolvent kernel

Fact

The resolvent kernel is built as follows:

- *First we solve the resolvent equation at level 1:*

$$\begin{cases} (\lambda - \Delta)\psi_p^{(\lambda)} = 0, & \text{on each } K_j = F_j(K), \\ \psi_p^{(\lambda)}(q) = \delta_{pq}, & \text{for } p \in V_1 \setminus V_0 \text{ and } q \in V_1. \end{cases}$$

Formula for the resolvent kernel

Fact

The resolvent kernel is built as follows:

- *First we solve the resolvent equation at level 1:*

$$\begin{cases} (\lambda - \Delta)\psi_p^{(\lambda)} = 0, & \text{on each } K_j = F_j(K), \\ \psi_p^{(\lambda)}(q) = \delta_{pq}, & \text{for } p \in V_1 \setminus V_0 \text{ and } q \in V_1. \end{cases}$$

- *Then we build a matrix $B^{(\lambda)}$ with the following entries*

$$B_{pq}^{(\lambda)} = \sum \partial_n \psi_p^{(\lambda)}(q).$$

The resolvent kernel (cont'd)

Fact

The resolvent kernel (cont'd)

Fact

- *If λ is not a Dirichlet eigenvalue, then $B^{(\lambda)}$ is an invertible matrix.*

The resolvent kernel (cont'd)

Fact

- *If λ is not a Dirichlet eigenvalue, then $B^{(\lambda)}$ is an invertible matrix.*
- *Let $G^{(\lambda)}$ be the inverse on $B^{(\lambda)}$.*

Fact

- *If λ is not a Dirichlet eigenvalue, then $B^{(\lambda)}$ is an invertible matrix.*
- *Let $G^{(\lambda)}$ be the inverse on $B^{(\lambda)}$.*
- *Define the map*

$$\Psi^{(\lambda)}(x,y) = \sum_{p,q \in V_1 \setminus V_0} G^{(\lambda)}_{pq} \psi^{(\lambda)}_p(x) \psi^{(\lambda)}_q(y).$$

Fact

- *If λ is not a Dirichlet eigenvalue, then $B^{(\lambda)}$ is an invertible matrix.*
- *Let $G^{(\lambda)}$ be the inverse on $B^{(\lambda)}$.*
- *Define the map*

$$\Psi^{(\lambda)}(x,y) = \sum_{p,q \in V_1 \setminus V_0} G^{(\lambda)}_{pq} \psi^{(\lambda)}_p(x) \psi^{(\lambda)}_q(y).$$

- *Finally, the resolvent kernel is given by*

$$R^{(\lambda)}(x,y) = \sum_{\omega} r_w \Psi^{(\frac{1}{N^m} r_w \lambda)}(F_w^{-1}x, F_w^{-1}y).$$

Example: Unit interval

Example

For the unit interval we have that

$$\psi^{(\lambda)}(x) = \frac{1}{\sinh \frac{\sqrt{\lambda}}{2}} \begin{cases} \sinh \sqrt{\lambda} x & x \leq \frac{1}{2} \\ \sinh \sqrt{\lambda}(1-x) & x \geq \frac{1}{2} \end{cases},$$

Example: Unit interval

Example

For the unit interval we have that

$$\psi^{(\lambda)}(x) = \frac{1}{\sinh \frac{\sqrt{\lambda}}{2}} \begin{cases} \sinh \sqrt{\lambda}x & x \leq \frac{1}{2} \\ \sinh \sqrt{\lambda}(1-x) & x \geq \frac{1}{2} \end{cases},$$

$$\Psi^{(\lambda)}(x,y) = \frac{\sinh \frac{\sqrt{\lambda}}{2}}{2\sqrt{\lambda}\cosh \frac{\sqrt{\lambda}}{2}} \psi^{(\lambda)}(x)\psi^{(\lambda)}(y),$$

Example: Unit interval

Example

For the unit interval we have that

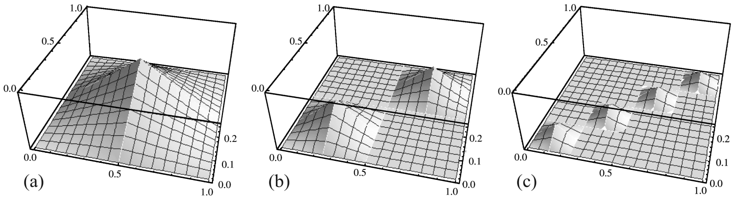
$$\psi^{(\lambda)}(x) = \frac{1}{\sinh \frac{\sqrt{\lambda}}{2}} \begin{cases} \sinh \sqrt{\lambda} x & x \leq \frac{1}{2} \\ \sinh \sqrt{\lambda}(1-x) & x \geq \frac{1}{2} \end{cases},$$

$$\Psi^{(\lambda)}(x,y) = \frac{\sinh \frac{\sqrt{\lambda}}{2}}{2\sqrt{\lambda} \cosh \frac{\sqrt{\lambda}}{2}} \psi^{(\lambda)}(x)\psi^{(\lambda)}(y),$$

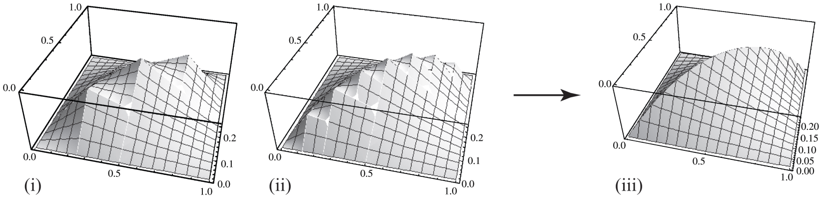
and

$$R^{(\lambda)}(x,y) = \sum_{m=0}^{\infty} \sum_{|\omega|=m} \frac{1}{2^m} \Psi^{(\lambda/4^m)}(F_\omega^{-1}x, F_\omega^{-1}y).$$

Unit interval: picture of $\psi^{(1)}(x, y)$



Unit interval: picture of $R^{(1)}(x, y)$



Example

For the Sierpinski gasket the matrix $G^{(\lambda)}$ is given by

$$G^{(\lambda)} = \frac{3}{5(5-\lambda_0)(2-\lambda_0)\tau(\lambda)} \begin{bmatrix} 3-\lambda_0 & 1 & 1 \\ 1 & 3-\lambda_0 & 1 \\ 1 & 1 & 3-\lambda_0 \end{bmatrix},$$

where

$$\tau(\lambda) = \frac{4\lambda}{3\lambda_0(2-\lambda_1)} \prod_{j=2}^{\infty} \left(1 - \frac{\lambda_j}{3}\right).$$