MATH 399, HOMEWORK #1

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Problem 1. If $f(x)$ is differentiable at $x = a$ and $f(a) \neq 0$, find

$$\lim_{n\to\infty} \left(\frac{f(a + \frac{1}{n})}{f(a)} \right)^n .$$

Problem 2. Let $X = [0, 1]$ and let $\Delta = \{(x, x) \in X \times X\}$ be the diagonal of the unit square. Assume that $f : X \times X \setminus \Delta \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are continuous, and that

$$\begin{aligned} \lim_{y\to x} f(x, y) &= g(x) \text{ and} \\ \lim_{x\to y} f(x, y) &= g(y). \end{aligned}$$

Dos it follow that f has a continuous extension to $X \times X$? If yes, prove it. If no, provide an example for which the statement fails.

Problem 3. Prove that if $a > 1$, then $\lim_{n\to\infty} \big(n^{-1} \log_a n\big) = 0$.

Problem 4. Evaluate

$$\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}}.$$

Hint: Find an integral formula for $\binom{2n}{n}^{-1}$.

Problem 5. Show that given a sequence of $n^2 + 1$ distinct integers, either there is an increasing subsequence of $n + 1$ terms or a decreasing subsequence of $n + 1$ terms.

Problem 6. The points of the plane are colored either red or blue. Prove that for some color there are pairs of points at every possible distance.

Problem 7. Prove that there exist irrational numbers a and b such that a^b is rational.

Problem 8. Let P be any point inside an equilateral triangle. Prove that the sum of the distances from P to each side of the triangle is equal to the height of the triangle.

Date: Due January 24, 2011.