This is a brief document about how to handle constants in some situations when working with differential equations (DE).

Ex: Consider the separable DE

$$\frac{dy}{dx} = 2yx.$$

We separate the variables to get

$$\frac{1}{y}\,dy = 2x\,dx.$$

Note here that $y$ cannot be zero, but $y$ could be zero in the original DE. (Also, $y = 0$ is a solution to the original DE since $y = 0$ and $y' = 0$ satisfy the DE.) We integrate both sides to get

$$\int \frac{1}{y}\,dy = \int 2x\,dx \quad \Rightarrow \quad \ln|y| = x^2 + c.$$

We can solve for $y$ has as follows:

$$\begin{aligned} & e^{\ln|y|} = e^{x^2+c} \\ \Rightarrow \quad & |y| = e^c e^{x^2} \\ \Rightarrow \quad & y = \pm e^c e^{x^2} \\ \Rightarrow \quad & y = Ke^{x^2}, \qquad \text{where } K = \pm e^c. \end{aligned}$$

But $K$ is never zero. Since $y = 0$ is also a solution to the original DE, we can now allow $K_1$ to be any real number, including zero. Our final solution is

$$y = K_1 e^{x^2}, \qquad \text{for } K_1 \in \mathbb{R}.$$

Ex: Now consider the separable initial value problem (IVP) DE

$$\frac{dy}{dx} = \frac{1}{y}, \qquad y(0) = 4$$

We separate the variables to get

$$y\,dy = 1\,dx$$

Note here that $y$ can be zero, but $y$ could not be zero in the original DE. We integrate both sides to get

$$\int y\,dy = \int dx \quad \Rightarrow \quad \frac{1}{2}y^2 = x + c' \quad \Rightarrow \quad y^2 = 2x + c \quad \text{with } c = 2c'.$$

Using our initial condition (IC) $y(0) = 4$, we have

$$(4)^2 = 2(0) + c \quad \Rightarrow \quad c = 16$$

so that our particular implicit solution is

$$y^2 = 2x + 16$$

We can solve for $y$ to get the explicit solution

$$y = \pm\sqrt{2x + 16},$$

and since we have the IC $y(0) = 4$, we know that our function is the positive curve, not the negative curve. Our function passes through the point $(0, 4)$. Thus, we have

$$y = \sqrt{2x + 16}.$$

Because of the IC, we can find this particular explicit solution. Without the IC, we are stuck with the implicit solution (because of the $\pm$).