A *first-order linear* differential equation (DE) can be put in the form

$$\frac{dy}{dx} + p(x)y = f(x) \qquad \text{(Standard Form)}$$

where both p and f are continuous functions. If a linear DE looks like

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \qquad a_1(x) \neq 0$$

then you must divide every term by $a_1(x)$ to rewrite the equation in standard form:

$$\frac{dy}{dx} + p(x)y = f(x) \qquad \text{where} \quad p(x) = \frac{a_0(x)}{a_1(x)} \quad \text{and} \quad f(x) = \frac{g(x)}{a_1(x)}.$$

Note: We must have the standard form before we can proceed to the next step.

For sake of argument, assume that exists a "magical" function $\mu(x)$, called an *integrating factor*, with the property that

$$\mu(x)p(x) = \mu'(x). \qquad \text{“magical property”}$$

If we have

$$\frac{dy}{dx} + p(x)y = f(x),$$

multiply every term by $\mu(x)$ to get

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)f(x).$$

Using the "magical property" of $\mu(x)$, the lefthand side becomes

$$\mu(x)\frac{dy}{dx} + \mu'(x)y = \mu(x)f(x).$$

Undoing the Product Rule on the lefthand side gives us

$$\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)f(x).$$

Integrating both sides with respect to x yields

$$\begin{aligned} \int \frac{d}{dx}\left[\mu(x)y\right]\,dx &= \int \mu(x)f(x)\,dx \\ \mu(x)y + c_1 &= \int \mu(x)f(x)\,dx \end{aligned}$$

Solving for y gives us the general solution

$$y = \frac{c_2 + \int \mu(x)f(x)\,dx}{\mu(x)} \qquad \text{where } c_2 = -c_1 \tag{1}$$

How do we get that magical function $\mu(x)$? We want it to satisfy the magical property

$$\mu(x)p(x) = \mu'(x).$$

This is a separable DE that can be written

$$\mu(x)p(x) = \frac{d\mu}{dx}.$$

Solving this, we have

$$\begin{aligned} \int \tfrac{1}{\mu(x)}\,d\mu &= \int p(x)\,dx && \text{(LHS wrt } \mu\text{. RHS wrt } x\text{.)} \\ \ln|\mu(x)| + k_1 &= \int p(x)\,dx && \text{where } k_1 \text{ is a constant} \\ \ln|\mu(x)| &= k_1 + \int p(x)\,dx && \text{where } k_2 = -k_1 \\ |\mu(x)| &= e^{k_1 + \int p(x)\,dx} \\ |\mu(x)| &= e^{k_1} e^{\int p(x)\,dx} \\ \mu(x) &= K e^{\int p(x)\,dx} && \text{where } K(= \pm e^{k_1}) \text{ is a constant} \end{aligned}$$

If we substitute this back into the general solution given in (1), we get

$$\begin{aligned} y &= \frac{c_2 + \int \mu(x) f(x)\,dx}{\mu(x)} \\ &= \frac{c_2 + \int \left(K e^{\int p(x)\,dx}\right) f(x)\,dx}{\left(K e^{\int p(x)\,dx}\right)} \\ &= \frac{\frac{c_2}{K} + \int \left(e^{\int p(x)\,dx}\right) f(x)\,dx}{\left(e^{\int p(x)\,dx}\right)}. \end{aligned}$$

Letting $C = c_2/K$, the solution to the first-order DE can be written as

$$y = \frac{C + \int \mu(x) f(x)\,dx}{\mu(x)}.$$

where

$$\mu(x) = e^{\int p(x)\,dx}$$

Note that we do not need a constant of integration for the integrating factor because the differential equation is unaffected by a constant multiple.

It is often easier to just run through the process than to just use the formula.

Here are the steps.

1. Before you start writing anything.
   - Make sure that you have a first-order linear differential equation.
   - Make sure it is in standard form
     $$\frac{dy}{dx} + p(x)y = f(x).$$
     Note the + in front of $p(x)y$.
2. Now compute the integrating factor
   $$\mu(x) = e^{\int p(x)\,dx}.$$
3. Multiply every term of your standard form equation by the integrating factor
   $$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)f(x).$$
4. Recalling that the "magical property" of $\mu(x)$, we have $\mu(x)p(x) = \mu'(x)$, and the Product Rule can be applied to the lefthand side; that is, our equation can be rewritten
   $$\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)f(x).$$
5. Integrate both sides of Step 4. If possible, solve for the solution of $y(x)$. Remember the constant of integration here.
6. If an initial condition is given, find the value for the constant of integration.

For a shorthand version, we have

Standard form.

$$\frac{dy}{dx} + p(x)y = f(x)$$

$$\Downarrow$$

Multiply by $\mu(x) = e^{\int p(x)\,dx}$

$$\Downarrow$$

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)f(x)$$

$$\Downarrow$$

$$\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)f(x)$$

$$\Downarrow$$

Integrate and find general solution.

$$\Downarrow$$

Use IC if given