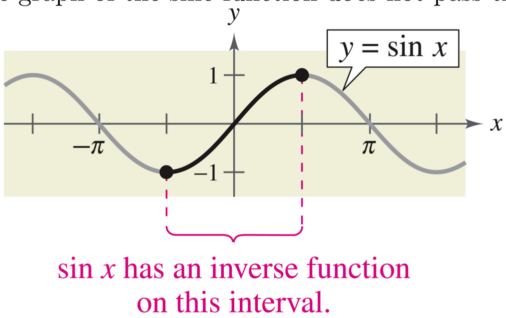
Inverse Trig Functions:

For a function to have an inverse, it must be one-to-one – that is, it must pass the Horizontal Line test. The graph of the sine function does not pass this test.



If we restrict our domain to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, we have:

1. On the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $y = \sin x$ is increasing.
2. On the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $y = \sin x$ takes on its full range of values, $-1 \le \sin x \le 1$.
3. On the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $y = \sin x$ is one-to-one.

Hence, on the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $y = \sin x$ has a unique inverse called the *inverse sine function*, denoted by

$$y = \arcsin x \qquad \text{or} \qquad y = \sin^{-1} x.$$

Note: The notation $\sin^{-1} x$ is consistent with the notation $f^{-1}(x)$; here,

$$\sin^{-1} x \neq \frac{1}{\sin x}.$$

The *inverse sine function* is defined by

$$y = \arcsin x \qquad \text{if and only if} \qquad \sin y = x,$$

where $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. The domain of $y = \arcsin x$ is $[-1, 1]$, and the range in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Ex: If possible, find the exact value of

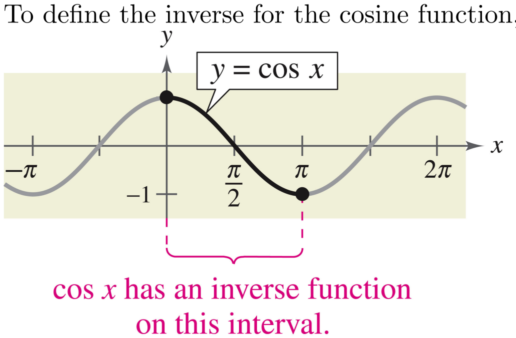
$$\arcsin\left(\frac{-\sqrt{2}}{2}\right) \qquad \sin^{-1}(-1) \qquad \arcsin 3$$

Since $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, we have $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

Since $\sin\left(-\frac{\pi}{2}\right) = -1$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, we have $\sin^{-1}(-1) = -\frac{\pi}{2}$.

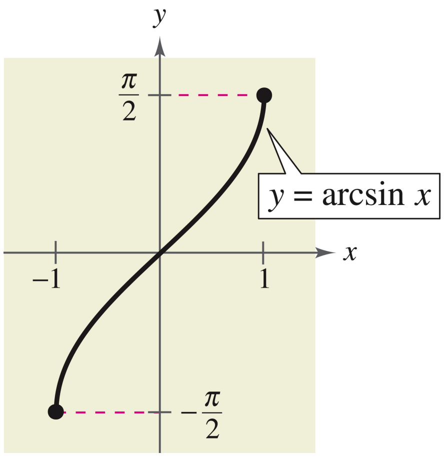
Since $-1 \le \sin x \le 1$ for all real numbers, $\arcsin 3$ has no solution.

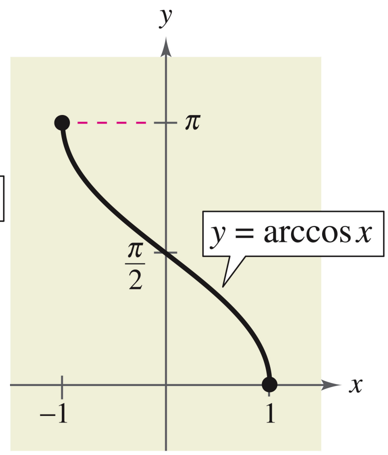
To define the inverse for the cosine function, we restrict our domain to $0 \le x \le \pi$.

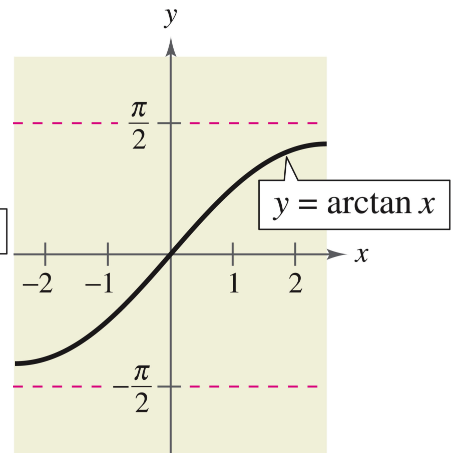


Definitions of the Inverse Trigonometric Functions:

Function	Domain	Range
$y = \arcsin x = \sin^{-1} x$ if and only if $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x = \cos^{-1} x$ if and only if $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x = \tan^{-1} x$ if and only if $\tan y = x$	$-\infty \le x \le \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$







To obtain these graphs, you can either follow the process on page 196 by filling out a table, plotting those points, and then filling in the curve, or you can reflect the graph of your function (with its restricted domain) across the line $y = x$.

Ex: $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$ since $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $0 \le \frac{5\pi}{6} \le \pi$.

Ex: $\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$ since $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$ and $-\frac{\pi}{2} \le \frac{\pi}{3} \le \frac{\pi}{2}$.

Inverse Property of Trigonometric Functions

If $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, then

$$\sin(\arcsin x) = x \qquad \text{and} \qquad \arcsin(\sin y) = y.$$

If $-1 \le x \le 1$ and $0 \le y \le \pi$, then

$$\cos(\arccos x) = x \qquad \text{and} \qquad \arccos(\cos y) = y.$$

If x is a real number and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, then

$$\tan(\arctan x) = x \qquad \text{and} \qquad \arctan(\tan y) = y.$$

Ex: If possible, find the exact value.

$$\cos^{-1}\left(\cos\left(-\frac{3\pi}{2}\right)\right) \qquad \sin\left(\arcsin\left(\frac{1}{2}\right)\right)$$

Note that $-\frac{3\pi}{2}$ is not in the interval $[0, \pi]$, but the coterminal angle $-\frac{3\pi}{2} + 2\pi = \frac{\pi}{2}$ is in the interval. So, we have

$$\cos^{-1}\left(\cos\left(-\frac{3\pi}{2}\right)\right) = \cos^{-1}\left(\cos\left(\frac{\pi}{2}\right)\right) = \frac{\pi}{2}.$$

Since $-1 \le \frac{1}{2} \le 1$, we have

$$\sin\left(\arcsin\left(\frac{1}{2}\right)\right) = \frac{1}{2}.$$

Ex: Find the exact values of

$$\tan\left(\arccos\left(\frac{1}{3}\right)\right) \qquad \sin\left(\arctan\left(-\frac{2}{3}\right)\right).$$

Pic. Let $u = \arccos\left(\frac{1}{3}\right)$ so that $\cos u = \frac{1}{3}$. We draw a triangle in the first quadrant since the angle is positive. We have hyp $= 3$ and adj $= 1$, and

$$(\text{adj})^2 + (\text{opp})^2 = (\text{hyp})^2 \quad \Rightarrow \quad \text{opp} = \sqrt{(\text{hyp})^2 - (\text{adj})^2} = \sqrt{(3)^2 - (1)^2} = 2\sqrt{2}.$$

So,

$$\tan\left(\arccos\left(\frac{1}{3}\right)\right) = \tan(u) = \frac{\text{opp}}{\text{adj}} = \frac{2\sqrt{2}}{1} = 2\sqrt{2}$$

Pic. Let $v = \arctan\left(-\frac{2}{3}\right)$ so that $\tan v = -\frac{2}{3}$. We draw our triangle in the fourth quadrant since $\tan u$ is negative. We have opp $= 2$ and adj $= 3$. Thus,

$$\text{hyp} = \sqrt{(\text{adj})^2 + (\text{opp})^2} = \sqrt{2^2 + 3^2} = \sqrt{13},$$

and

$$\sin\left(\arctan\left(-\frac{2}{3}\right)\right) = \sin(v) = \frac{\text{opp}}{\text{hyp}} = -\frac{2}{\sqrt{13}}$$

Note that its negative since we are in Quadrant IV.

Ex: Write $\sec\left(\sin^{-1} 7x\right)$, with $0 \le x \le \frac{1}{7}$ (to avoid complex numbers), as an algebraic expression.

Let $u = \sin^{-1} 7x$ so that $\sin u = 7x$, where $-1 \le 7x \le 1$. We have opp $= 7x$ and hyp $= 1$. We draw our triangle in Quadrant I with u an acute angle since $\sin u = \frac{\text{opp}}{\text{hyp}} = \frac{7x}{1}$. Thus,

$$(\text{adj})^2 + (\text{opp})^2 = (\text{hyp})^2 \quad \Rightarrow \quad \text{adj} = \sqrt{(\text{hyp})^2 - (\text{opp})^2} = \sqrt{(1)^2 - (7x)^2} = \sqrt{1 - 49x^2}.$$

Thus,

$$\sec\left(\sin^{-1} 7x\right) = \sec(u) = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\sqrt{1 - 49x^2}}.$$

The remaining inverse trigonometric functions are not used as frequently and are summarized here.

$$y = \csc^{-1}x \;\; (|x| \geqslant 1) \quad \Longleftrightarrow \quad \csc y = x \quad \text{and} \quad y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$

$$y = \sec^{-1}x \;\; (|x| \geqslant 1) \quad \Longleftrightarrow \quad \sec y = x \quad \text{and} \quad y \in [0, \pi/2) \cup [\pi, 3\pi/2)$$

$$y = \cot^{-1}x \;\; (x \in \mathbb{R}) \quad \Longleftrightarrow \quad \cot y = x \quad \text{and} \quad y \in (0, \pi)$$

The choice of intervals for y in the definitions of $\csc^{-1}$ and $\sec^{-1}$ is not universally agreed upon. For instance, some authors use $y \in [0, \pi/2) \cup (\pi/2, \pi]$ in the definition of $\sec^{-1}$.