Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational fractional function where $P(x)$ and $Q(x)$ have no common factors and $Q(x) \neq 0$. We will rewrite $f(x)$ as sum of simpler rational functions called *partial fractions*. We have four steps.

Step 1: Use long division if you have an *improper rational function* (the degree of the numerator $\geq$ than the degree of the denominator).

$$\frac{P(x)}{Q(x)} = \text{(a polynomial)} + \frac{R(x)}{Q(x)}$$

If you have a *proper rational function* (the degree of the numerator $<$ than the degree of the denominator), then you are ready for Step 2. (In this case, $R(x) = P(x)$ from above and the polynomial is simply zero.)

Step 2: Factor the denominator completely into a product of linear and/or irreducible quadratic factors with real coefficients.

Step 3: Rewrite the original fraction into partial fractions using the following forms:

Case 1: $Q(x)$ is a product of distinct linear factors with no repeated factors.

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n)$$

We write

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nx + b_n}$$

where $A_1, A_2, \ldots, A_n$ are constants to be found later.

Ex:

$$\frac{3x^2 + 7x - 2}{x^3 - x^2 - 2x} = \frac{3x^2 + 7x - 2}{x(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

Case 2: $Q(x)$ is a product of linear factors, where some factors may be repeated; that is, $(a_1x + b_1)^r$ occurs in that factorization of $Q(x)$.

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

Here, $A_1, A_2, \ldots, A_r$ are constants to be found later.

Ex:

$$\frac{2x^2 + 5x + 1}{x^2(x+2)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2} + \frac{E}{(x+2)^3}$$

Case 3: $Q(x)$ contains irreducible quadratic factors, none of which are repeated. Such a quadratic factor $ax^2 + bx + c$ of $Q(x)$ will have $b^2 - 4ac < 0$, and we will have a corresponding fraction $\dfrac{Ax + B}{ax^2 + bx + c}$, where A and B will be found later.

Ex:

$$\frac{x - 1}{(x+2)(x^2+1)(x^2+x+1)} = \frac{A}{x+2} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{x^2 + x + 1}$$

Case 4: $Q(x)$ contains repeated irreducible quadratic factors. Such a quadratic factor $\left(ax^2 + bx + c\right)^r$ of $Q(x)$ will have $b^2 - 4ac < 0$, and we will have the corresponding expression

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r},$$

where each A_i and B_i will be found later.

Ex:

$$\frac{x^2 - 7}{x(x^2+2)^3} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2} + \frac{Dx + E}{(x^2+2)^2} + \frac{Fx + G}{(x^2+2)^3}$$

Step 4: Determine the constants A, B, C *etc.* using one of the following two methods.

Method 1: Multiply by the least common denominator (LCD) and equate the coefficients of like terms.

Ex: $\dfrac{6x^2-3x+1}{(4x+1)(x^2+1)} = \dfrac{A}{4x+1} + \dfrac{Bx+C}{x^2+1}$

1. Multiply both sides by the LCD $= (4x+1)\left(x^2+1\right)$ to get

$$6x^2-3x+1 = A\left(x^2+1\right) + (Bx+C)(4x+1)$$

2. Expand the righthand side and combine like terms.

$$6x^2-3x+1 \quad = \quad (A+4B)x^2 + (B+4C)x + A + C$$

3. Set corresponding coefficients equal and solve for constants A, B, C, D, E.

$$\begin{aligned} 6 &= A+4B && \text{Coefficients of } x^2 \\ -3 &= B+4C && \text{Coefficients of } x \\ 1 &= A+C && \text{Constant terms} \end{aligned}$$

Solving the third equation for A yields $A = 1-C$. We rewrite the first equation as

$$6 = A+4B \quad \Rightarrow \quad 6 = 1-C+4B \quad \Rightarrow \quad C = -5+4B.$$

Now using the second equation, we have

$$-3 = B+4C \quad \Rightarrow \quad -3 = B+4(-5+4B) \quad \Rightarrow \quad B = 1$$

With this, we see that $C = -5+4B = -5+4(1) = -1$ and $A = 1-C = 1-(-1) = 2$.
Therefore, we have

$$\frac{6x^2-3x+1}{(4x+1)(x^2+1)} = \frac{2}{4x+1} + \frac{x-1}{x^2+1}$$

Method 2: Solve by Substitution. Find appropriate values for x that simplify the equation and allow you to find the values for A, B, C *etc*.

Ex: $\dfrac{6x^2-3x+1}{(4x+1)(x^2+1)} = \dfrac{A}{4x+1} + \dfrac{Bx+C}{x^2+1}$

1. Multiply both sides by the LCD $= (4x+1)\left(x^2+1\right)$ to get

$$6x^2-3x+1 = A\left(x^2+1\right) + (Bx+C)(4x+1) \tag{1}$$

2. If we choose $x = -\frac{1}{4}$, Equation (1) above becomes

$$\frac{6}{16} + \frac{3}{4} + 1 = A\left(\frac{17}{16}\right) \quad \Rightarrow \quad A = 2$$

If we choose $x = 0$, Equation (1) above becomes

$$1 = A+C = 2+C \quad \Rightarrow \quad C = -1$$

If we choose $x = 1$, Equation (1) above becomes

$$4 = A(2) + (B+C)(5) = 4 + 5(B-1) \quad \Rightarrow \quad B = 1$$

Therefore, we have

$$\frac{6x^2-3x+1}{(4x+1)(x^2+1)} = \frac{2}{4x+1} + \frac{x-1}{x^2+1}$$