Ex: $\int \cos^2\ dx$

Letting $u = \cos x$ and $dv = \cos\ dx$, we have $du = -\sin x\, dx$ and $v = \sin x$. Thus,

$$\begin{aligned}
\int \cos^2\ dx &= \cos x \sin x - \int \sin x\,(-\sin x)\ dx \\
&= \cos x \sin x + \int \sin^2 x\, dx \\
&= \cos x \sin x + \int \left(1 - \cos^2 x\right)\ dx \\
&= \cos x \sin x + \int 1\, dx - \int \cos^2 x\, dx \\
&= \cos x \sin x + x + c - \int \cos^2 x\, dx
\end{aligned}$$

Notice here that we ended up back where we started, with $\int \cos^2\ dx$. So we add that integral to both sides to get

$$2\int \cos^2\ dx = \cos x \sin x + x + c$$

Now divide both sides by 2 to get

$$\int \cos^2\ dx = \frac{1}{2}\cos x \sin x + \frac{x}{2} + c'$$

where $c' = c/2$.

Ex: $\int e^x \sin x \, dx$

Solution 1: Using integration by parts, we have

$$\begin{aligned}
\int e^t \sin t \, dt &= -e^t \cos t - \int (-\cos t) \, e^t \, dt \\
&\quad \text{with } \begin{bmatrix} u = e^t & dv = \sin t \, dt \\ du = e^t \, dt & v = -\cos t \end{bmatrix} \\
&= -e^t \cos t + \int e^t \cos t \, dt \\
&= -e^t \cos t + \left(e^t \sin t - \int (\sin t) \, e^t \, dt \right) \\
&\quad \text{with } \begin{bmatrix} U = e^t & dV = \cos t \, dt \\ dU = e^t \, dt & V = \sin t \end{bmatrix} \\
&= -e^t \cos t + e^t \sin t - \int e^t \sin t \, dt
\end{aligned}$$

so that

$$\int e^t \sin t \, dt = -e^t \cos t + e^t \sin t - \int e^t \sin t \, dt.$$

Moving the integrals to the lefthand side, we have

$$2 \int e^t \sin t \, dt = -e^t \cos t + e^t \sin t + c.$$

Now, dividing by the constant 2, we have our final answer:

$$\int e^t \sin t \, dt = -\frac{1}{2} e^t \cos t + \frac{1}{2} e^t \sin t + c',$$

where $c' = c/2$.

Question: What would have happen if we initially chose $u = \sin x$ and $dv = e^x \, dx$?

See next page for comments on this example.

Note: If you pick a u and dv, and have to use integration by parts a second time, be consistent and stick with a similar choice of u and dv. Below is what happens if you switch u and dv on the second use of integration of parts.

$$
\begin{aligned}
\int e^t \sin t \, dt &= -e^t \cos t - \int (-\cos t)\, e^t \, dt \\
&\quad \text{with } \begin{bmatrix} u = e^t & dv = \sin t \, dt \\ du = e^t \, dt & v = -\cos t \end{bmatrix} \\
&= -e^t \cos t + \int e^t \cos t \, dt \\
&= -e^t \cos t + \left(e^t \cos t - \int e^t \left(-\sin t\right) \, dt \right) \\
&\quad \text{with } \begin{bmatrix} U = \cos t & dV = e^t \, dt \\ dU = -\sin t \, dt & V = e^t \end{bmatrix} \\
&\quad \text{Notice that } e^t \text{ is now with } dV \text{ instead of staying with } U. \\
&= -e^t \cos t + e^t \cos t + \int e^t \sin t \, dt \\
&= \int e^t \sin t \, dt
\end{aligned}
$$

This is back where we started. So, if you choose $u = e^t$ and $dv = \sin t \, dt$ the first time you use integration of parts, be consistent and choose $U = e^t$ and $dV = \cos t \, dt$ the second time you use integration by parts.