Approximation of Local Fields in Nonlinear Power Law Materials

Silvia Jimenez* and Robert Lipton

Department of Mathematics, Louisiana State University, Baton Rouge

{sjimenez,lipton}@math.lsu.edu

Abstract

Properties of local fields inside mixtures of two nonlinear power law materials are studied. This simple constitutive model is frequently used to describe several phenomena ranging from plasticity to optical nonlinearities in dielectric media. This work addresses a prototypical problem in the scalar setting. We provide the corrector theory for the strong approximation of fields inside composites made from power law materials. These results are applied to deliver new multiscale tools for bounding the local singularity strength inside micro-structured media in terms of the macroscopic applied fields.

1. Notation and Statement of the Problem

Since the mixture is periodic, the unit period cell Y is used to define χ_1^ϵ and χ_2^ϵ. Let F be an open subset of Y of material one, with smooth boundary ∂F, such that $\overline{F} \subset Y$. The function $\chi_1(y) = 1$ inside F and 0 outside and $\chi_2(y) = 1 - \chi_1(y)$. Denote by $\overline{F}_i^\epsilon$ the set of all translated images $\epsilon\left(i + \overline{F}\right)$ of $\epsilon\overline{F}$ by the vector ϵi, $i \in \mathbb{Z}^n$. Then the ϵ-periodic mixture inside Ω is described by

$$\chi_1^\epsilon(x) = \chi_1(x/\epsilon) \text{ and } \chi_2^\epsilon(x) = \chi_2(x/\epsilon).$$

This kind of microstructure made of inclusions of one material surrounded by the other material phase is often referred as disperse microstructure (See Figure 1).

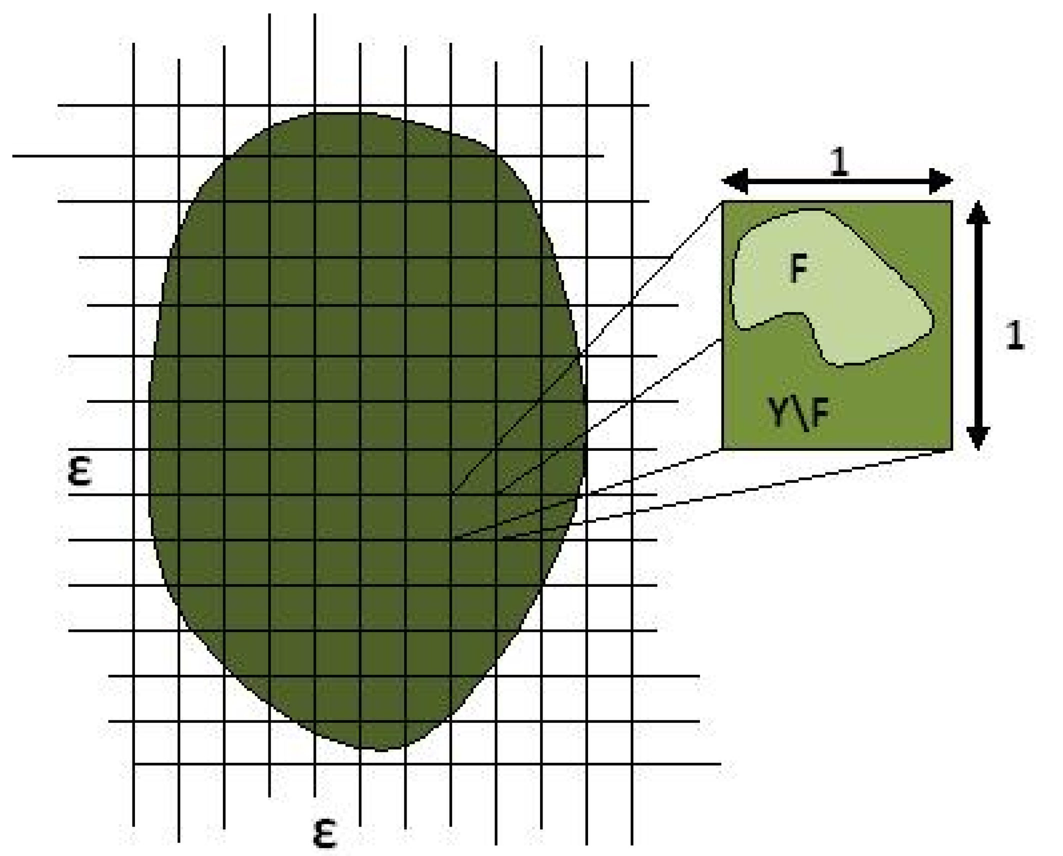


Figure 1: *Disperse Microstructure*

Results in the case of layered materials are also obtained. In this case the representative unit cell consists of a layer of material one, denoted by R_1, sandwiched between layers of material two, denoted by R_2. The interior boundary of R_1 is denoted by Γ. Here $\chi_1(y) = 1$ for $y \in R_1$ and 0 in R_2, and $\chi_2(y) = 1 - \chi_1(y)$ (See Figure 2).

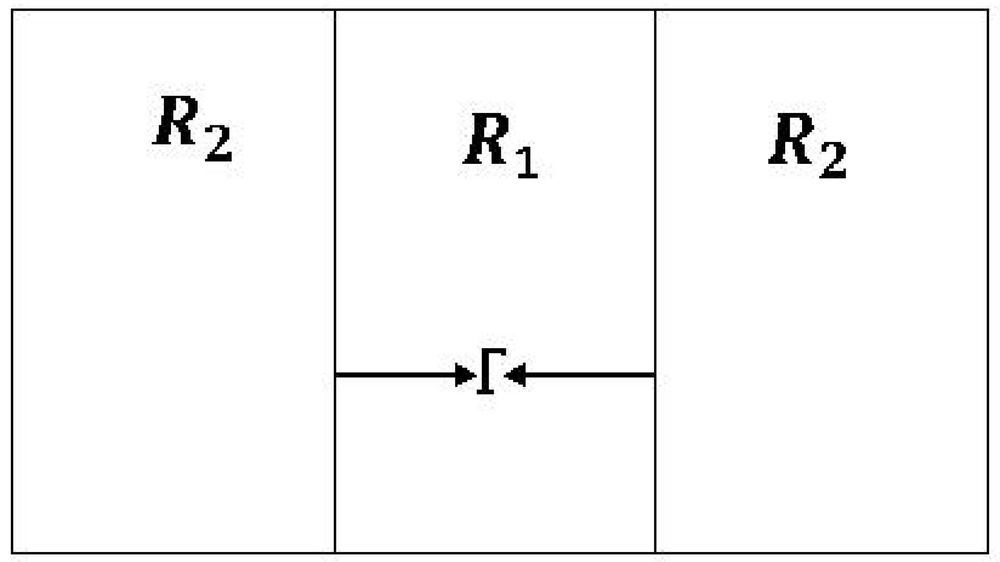


Figure 2: *Layered Microstructure*

The exponents for material one and material two are denoted p_1 and p_2 respectively and satisfy $2 \le p_1 \le p_2$, and their Hölder conjugates are denoted by q_2 and q_1 respectively.

The piecewise power law material is defined by the constitutive law $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$A(x,\xi) = \sigma(x)\,|\xi|^{p(x)-2}\,\xi = \sigma_1\chi_1(x)\,|\xi|^{p_1-2}\,\xi + \sigma_2\chi_2(x)\,|\xi|^{p_2-2}\,\xi$$

with

$$\sigma(x) = \chi_1(x)\,\sigma_1 + \chi_2(x)\,\sigma_2, \text{ and } p(x) = \chi_1(x)\,p_1 + \chi_2(x)\,p_2,$$

and the constitutive law for the ϵ-periodic composite is given by

$$A_\epsilon(x,\xi) = A\left(\frac{x}{\epsilon}, \xi\right), \text{ for every } \epsilon > 0,$$

for every $x \in \mathbb{R}^n$ and for every $\xi \in \mathbb{R}^n$.

2. Homogenization Theory

2.1 Dirichlet Problem

$$\begin{cases} -div\left(A_\epsilon\left(x, \nabla u_\epsilon\right)\right) = f \text{ on } \Omega, \\ u_\epsilon \in W_0^{1,p_1}(\Omega); \end{cases} \tag{1}$$

where $f \in W^{-1,q_2}(\Omega)$.

The differential operator appearing on the left hand side of (1) is commonly referred to as the $p_\epsilon(x)$-Laplacian. For the case at hand the exponents $p(x)$ and coefficients $\sigma(x)$ are taken to be simple functions. Because the level sets associated with these functions can be quite general and irregular they are referred to as rough exponents and coefficients. In this context all solutions are understood in the usual weak sense.

2.2 Homogenization Theorem

As $\epsilon \to 0$, the solutions u_ϵ of (1) converge weakly to u in $W^{1,p_1}(\Omega)$, where u is the solution of

$$\begin{cases} -div\left(b\left(\nabla u\right)\right) = f \text{ on } \Omega, \\ u \in W_0^{1,p_1}(\Omega); \end{cases} \tag{2}$$

and the function $b : \mathbb{R}^n \to \mathbb{R}^n$ is defined for all $\xi \in \mathbb{R}^n$ by

$$b(\xi) = \int_Y A(y, p(y,\xi))dy, \tag{3}$$

where $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$p(x,\xi) = \xi + \nabla v_\xi(x), \tag{4}$$

where v_ξ is the solution to the cell problem:

$$\begin{cases} \int_Y \left(A(y, \xi + \nabla v_\xi), \nabla w\right) dy = 0 \text{ for every } w \in W_{per}^{1,p_1}(Y), \\ v_\xi \in W_{per}^{1,p_1}(Y). \end{cases} \tag{5}$$

For $i = 1, 2$, $W_{per}^{1,p_i}(Y)$ denotes the set of all functions $u \in W^{1,p_i}(Y)$ with mean value zero which have the same trace on the opposite faces of Y.

3. Regularity Theory

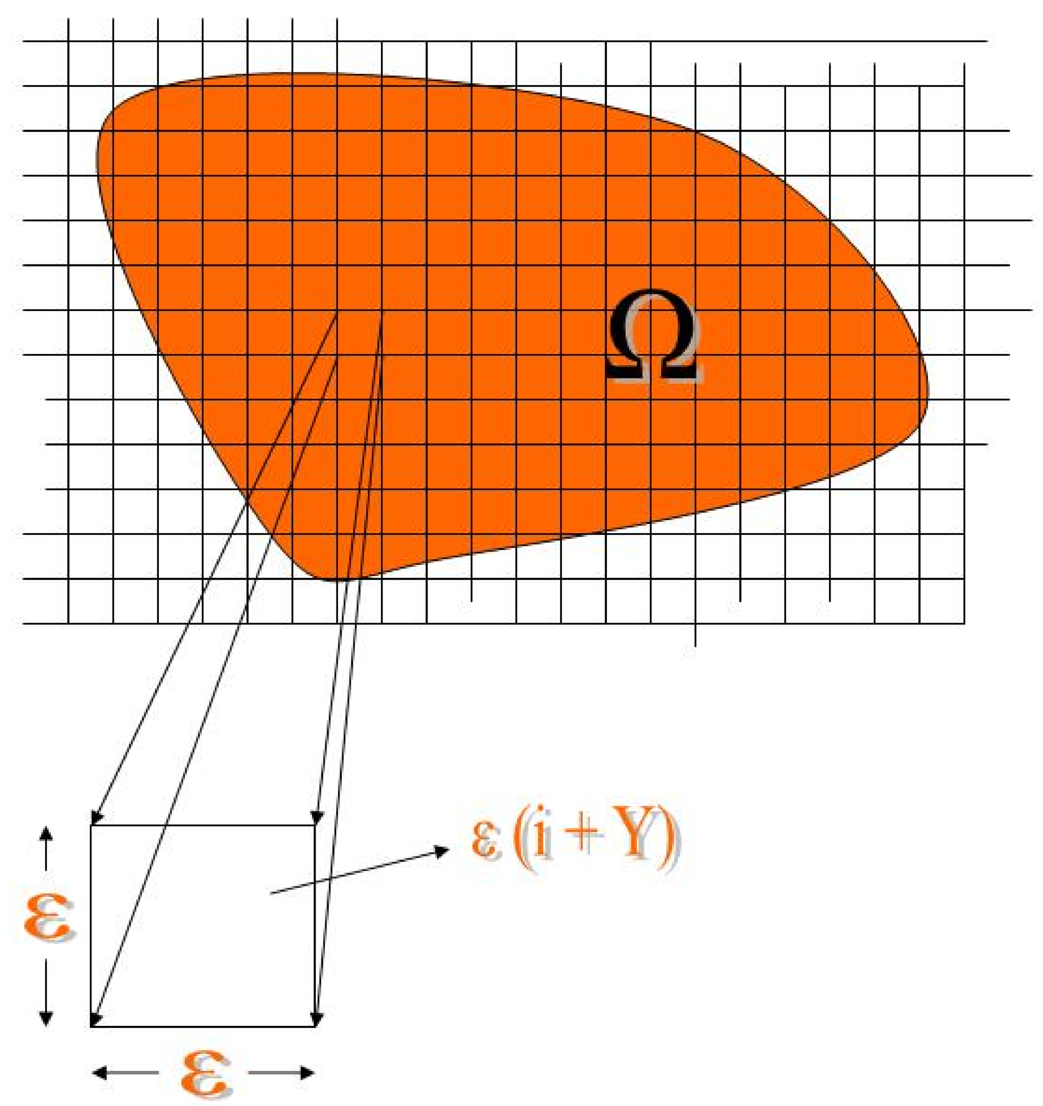
If we assume that we have a periodic disperse structure or a layered material, then the solution u of (2) belongs to $W_0^{1,p_2}(\Omega)$.

4. Statement of the Corrector Theorem

In this section, we construct a family of correctors that approximate the sequence $\left\{\chi_i^\epsilon \nabla u_\epsilon\right\}_{\epsilon>0}$ strongly in $L^{p_i}(\Omega, \mathbb{R}^n)$.

4.1 Notation

- $Y_\epsilon^i = \epsilon(i + Y)$, where $i \in \mathbb{Z}^n$ and $\epsilon > 0$, i.e. the translated image of $Y_\epsilon = \epsilon Y$ by the vector ϵi.
- $I_\epsilon = \left\{i \in \mathbb{Z}^n : Y_\epsilon^i \subset \Omega\right\}$.



Let $\varphi \in L^{p_2}(\Omega;\mathbb{R}^n)$ and $M_\epsilon\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be the function defined by

$$M_\epsilon(\varphi)(x) = \sum_{i\in I_\epsilon} \chi_{Y_\epsilon^i}(x)\frac{1}{|Y_\epsilon^i|}\int_{Y_\epsilon^i}\varphi(y)dy, \tag{6}$$

4.2 Corrector Theorem

Let $f \in W^{-1,q_2}(\Omega)$, let u_ϵ be the solutions to the problem (1), and let u be the solution to problem (2). Then, for periodic disperse structures and layered materials, we have

$$\int_\Omega \left|\chi_i^\epsilon p_\epsilon\left(x, M_\epsilon(\nabla u)\right) - \chi_i^\epsilon \nabla u_\epsilon\right|^{p_i} \to 0, \text{ as } \epsilon \to 0. \tag{7}$$

for $i = 1, 2$.

5. Fluctuations Result

For all Caratheodory functions $\psi \ge 0$ and measurable sets $D \subset \Omega$ we have

$$\int_D \int_Y \psi\left(y, \chi_i(y) p\left(y, \nabla u(x)\right)\right) dy dx \le \liminf_{\epsilon\to 0} \int_D \psi\left(x, \chi_i^\epsilon(x) \nabla u_\epsilon(x)\right) dx.$$

If the sequence $\left\{\psi\left(x, \chi_i^\epsilon(x)\nabla u_\epsilon(x)\right)\right\}$ is weakly convergent in $L^1(\Omega)$, then the inequality becomes an equality.

In particular,

$$\int_D \int_Y \chi_i(y) \left|p\left(y, \nabla u(x)\right)\right|^r dy dx \le \liminf_{\epsilon\to 0} \int_D \chi_i^\epsilon(x) \left|\nabla u_\epsilon(x)\right|^r dx,$$

for $r > 1$.

References

1. Dal Maso, G. and Defranceschi, A., "Correctors for the Homogenization of Monotone Operators", Differential and Integral Equations, Volume 3, Number 6, November 1990, pp.1151–1166.
2. Jimenez S. and Lipton R., "Correctors and Field Fluctuations for the $p_\epsilon(x)$-Laplacian with Rough Exponents". In preparation.
3. Lipton, R., "Homogenization and Field Concentrations in Heterogeneous Media", SIAM Journal on Mathematical Analysis, 38, 2006, pp.1048–1059.
4. Pedregal,P., "Parametrized Measures and Variational Principles", Birkhauser Verlag, Boston, 1997.
5. Jikov, V.V. et al, "Homogenization of Differential Operators and Integral Functions", Springer-Verlag, 1994.