Nonlinear Composites

Silvia Jiménez
Department of Mathematics
Louisiana State University

August 20, 2009
Research Groups and People

- Analysis Research Group at LSU
- Partial Differential Equations Group
- Scientific Computing and Numerical Analysis
- Research Group in Materials Science at LSU
Outline

- Introduction
- What is the goal and what do we need?
- Possible Courses
- Mixtures of N Power Law Materials with Same Exponent
- Homogenization Theorem (Fusco and Moscariello)
- Corrector Theorem (Dal Maso and Defranceschi)
- Lower bound on Field Concentrations
Heterogeneous materials are abundant in nature, and increasingly so in manmade systems.
Often the distribution of heterogeneity is such that the material appears to be *homogeneous* at a large enough length scale.

What are the "average" properties of such heterogeneous materials?

"Effective properties" of such "composite materials," directly from the properties of their "microstructure".
In my research, I work with composites with nonlinear constitutive behaviour, called power law materials.

Used to describe several phenomena ranging from plasticity to optical nonlinearities in dielectric media.
Introduction

- Important to provide insight into the statistical distribution of local quantities, such as averages or higher moments of the strains and stresses in each phase.
- These local quantities are extremely useful for understanding the evolution of nonlinear phenomena such as plasticity or damage.
Research on composite materials with nonlinear constitutive behaviour can be traced back to the classical work of Taylor (“Plastic Strain in Metals”, 1938) on the plasticity of polycrystals.
In composites, failure initiation is a multiscale phenomena. A load applied at the structural scale is often amplified by the microstructure creating local zones of high field concentration. Properties of local fields inside mixtures of two nonlinear power law materials are studied. This work addresses a prototypical problem in the scalar setting.
What is the goal and what do we need?

- The goal is to develop new multiscale tools to bound the local singularity strength inside micro-structured media in terms of the macroscopic applied fields.
- The research carried out in this project draws upon the mathematical theory of Elliptic PDEs, Corrector theory, Young measures, and Homogenization methods.
Possible courses

- MATH 4999-5: Vertically Integrated Research: Complex Materials and Fluids. Instructor: Prof. Lipton with S. Armstrong: Introduction to the field of homogenization theory, guide to the current research literature useful for understanding the mathematics and physics of complex heterogeneous media.


- 7380-3 Partial Differential Equations. Prof. Shipman.: Wide variety of topics in partial differential equations with a detailed understanding of illuminating examples.

- SPRING 2010: 7390 Topic TBA in Materials Science. Prof. Lipton
We consider $N$ nonlinear power law materials periodically distributed inside a domain $\Omega$.

$\Omega$ is an open bounded subset of $\mathbb{R}^n$

The length scale of the microstructure relative to the domain is denoted by $\epsilon$.

We describe the geometry of the mixture through the characteristic functions $\chi_i^\epsilon, i = 1, \ldots, N$, corresponding to each of the materials.

For $i = 1, \ldots, N$, $\chi_i^\epsilon = 1$ in the $i$-th phase and zero outside.
Mixtures of N Power Law Materials with Same Exponent

- Since the mixture is periodic, we use the unit period cell $Y$ to define $\chi_i^\epsilon$, $i = 1, \ldots, N$.
- For $i = 1, \ldots, N$, the indicator function of the $i$-th phase in the unit cell $Y$ is $\chi_i(y)$.
- $\sum_{i=1}^{N} \chi_i(y) = 1$.
- The $\epsilon$ periodic mixture inside $\Omega$ is described by

$$\chi_i^\epsilon(x) = \chi_i(x/\epsilon) \text{ for } i = 1, \ldots, N.$$
Let $p \geq 2$ and let $q$ such that
\[
\frac{1}{p} + \frac{1}{q} = 1.
\]

The piecewise power law material is defined by the constitutive law $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by
\[
A(y, \xi) = \sum_{i=1}^{N} \chi_i(y) a_i |\xi|^{p-2} \xi, \text{ with } a_i \geq 0,
\]
and the constitutive law for the $\epsilon$-periodic composite is given by
\[
A_\epsilon(x, \xi) = A \left( \frac{x}{\epsilon}, \xi \right), \text{ for every } \epsilon > 0.
\]
For a given source term $f \in W^{-1,q}(\Omega)$, consider the Dirichlet problem

\[
\begin{cases}
-\text{div} \left( A_\epsilon (x, \nabla u_\epsilon) \right) = f \quad \text{on } \Omega, \\
u_\epsilon \in W^{1,p}_0(\Omega).
\end{cases}
\]

where

\[A_\epsilon(x, \xi) = \sum_{i=1}^{N} \chi_\epsilon^i(x) a_i |\xi|^{p-2} \xi\]

and

\[
\frac{1}{p} + \frac{1}{q} = 1.
\]
As $\epsilon \to 0$, the solutions $u_\epsilon$ converge strongly to $u$ in $L^p(\Omega)$, and $\nabla u_\epsilon$ converges weakly to $\nabla u$ in $L^p(\Omega; \mathbb{R}^n)$, where $u$ is solution of

$$
\begin{cases}
-\text{div} (b(\nabla u)) = f \text{ on } \Omega, \\
u \in W^{1,p}_0(\Omega);
\end{cases}
$$

where $b : \mathbb{R}^n \to \mathbb{R}^n$ is defined for all $\xi \in \mathbb{R}^n$ by

$$
b(\xi) = \int_Y A(y, p(y, \xi))dy,
$$

where $p(y, \xi) = \xi + \nabla \upsilon(y)$, where $\upsilon$ is the solution to the cell problem:

$$
\begin{cases}
\int_Y (A(y, \xi + \nabla \upsilon), \nabla w) dy = 0 \text{ for every } w \in W^{1,p}_{\text{per}}(Y), \\
\upsilon \in W^{1,p}_{\text{per}}(Y).
\end{cases}
$$
\( W^{1,p}_{\text{per}}(Y) \) denotes the set of all functions \( u \in W^{1,p}(Y) \) with mean value zero which have the same trace on the opposite faces of \( Y \).

As before, for \( \epsilon > 0 \), we rescale and define

\[
p_\epsilon(x, \xi) = p \left( \frac{x}{\epsilon}, \xi \right) = \xi + \nabla u_\xi \left( \frac{x}{\epsilon} \right).
\]
Mixtures of N Power Law Materials with Same Exponent - Corrector Theorem (Dal Maso and Defranceschi)

- $Y^i_\epsilon = \epsilon(i + Y)$, where $i \in \mathbb{Z}^n$.
- $I_\epsilon = \{i \in \mathbb{Z}^n : Y^i_\epsilon \subset \Omega\}$.
- Let $\varphi \in L^p(\Omega, \mathbb{R}^n)$ and $M_\epsilon \varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a function defined by
  
  $$M_\epsilon(\varphi)(x) = \sum_{i \in I_\epsilon} \chi_{Y^i_\epsilon}(x) \frac{1}{|Y^i_\epsilon|} \int_{Y^i_\epsilon} \varphi(y) dy.$$

  $M_\epsilon$ takes the average of the vector field in every cube.
Mixtures of N Power Law Materials with Same Exponent - Corrector Theorem (Dal Maso and Defranceschi)

Theorem (Corrector Theorem)

\[ \left\| p \left( \frac{x}{\varepsilon}, M_\varepsilon(\nabla u)(x) \right) - \nabla u_\varepsilon(x) \right\|_{L^p(\Omega, \mathbb{R}^n)} \to 0, \text{ as } \varepsilon \to 0. \]
We use the Corrector Theorem and Young Measures to study the behavior of gradients of solutions ($\nabla u_\epsilon$) of the Dirichlet problem.

These tools allow us to bound nonlinear quantities of the gradients from below in terms of the local solution $p$ and the homogenized gradient.

Numerical Methods for the Homogenized correctors are simpler than the original $\epsilon$ problem. This is a multiscale lower bound.
Mixtures of N Power Law Materials with Same Exponent - Lower bound on Field Concentrations

**Theorem (Lower Bound of Field Concentrations)**

We have that for all Caratheodory function $\psi \geq 0$ and $E \subset \Omega$ measurable

$$\int_D \int_Y \psi(x, (p(y, \nabla u(x)))) dy dx \leq \liminf_{\epsilon \to 0} \int_D \psi(x, \nabla u_\epsilon(x)) dx.$$  

- In particular, for $r > 1$,

$$\int_D \int_Y |p(y, \nabla u(x))|^r dy dx \leq \liminf_{\epsilon \to 0} \int_D |\nabla u_\epsilon(x)|^r dx.$$  

- Functions of the form $\psi$ are often used in failure criteria (Tsai-Hahn/Tsai-Hill/Tsai-Wu).

- If the sequence $\psi(x, \nabla u_\epsilon(x))$ is weakly convergent in $L^1(\Omega)$, then the inequality becomes an equality.
In particular, for $r > 1$,

$$\int_D \int_Y |p(y, \nabla u(x))|^r \, dy \, dx \leq \liminf_{\epsilon \to 0} \int_D |\nabla u_\epsilon(x)|^r \, dx.$$ 

A use of this inequality is found in the Weibull Theory of Failure.

Probability of failure in $D = 1 - e^{-c \int_D |\nabla u_\epsilon|^r \, dx}$,

where $r= \text{scatter}$, which is a material property. So we get an lower bound for this quantity.

Thank you!