Corrector Theory for the Homogenization of Nonlinear Composite Materials

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July 9th, 2008
Outline

- Introduction/N-Phase Power Law Materials/Notation/

Preliminary Results-Homogenization Theory

What do we want to get?

Preliminary Results-Corrector Theory

Lower bound on Field Concentrations

Mix of 2 Different Power-Law Materials/Homogenization Theory

Corrector Theorem.

Future work.
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- Introduction/N-Phase Power Law Materials/Notation/
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- Corrector Theorem.
- Future work.
I would like to thank Prof. Robert Lipton for his help and advice.
Composites are materials made from two or more constituent materials with significantly different physical or chemical properties and which remain separate and distinct on a macroscopic level within the finished structure.
Introduction - Examples of Composite Materials

- Toughened Graphite
- Graphite
- Hybrid
- Fiberglass

Boeing 777
Physical parameters -conductivity, elasticity coefficients, etc- are discontinuous and change values between components across a small length scale $\epsilon$.

**Figure:** Heterogeneous Material
Physical parameters - conductivity, elasticity coefficients, etc. - are discontinuous and change values between components across a small length scale $\epsilon$. When components are intimately mixed, the physical parameters oscillate rapidly and microscopic structure becomes complicated.

Figure: Heterogeneous Material
**Important problem:** Determination of macroscopic properties of heterogeneous materials.
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A good approximation to the macroscopic behavior of such materials is obtained through a suitable asymptotic theory called Homogenization Theory.

Homogenization Theory provides an accurate description of the macroscopic properties as the length scale $\epsilon$ tends to zero in the equations describing phenomena such as heat conduction or elasticity.
A good model for the study of physical behaviour of heterogeneous material is given by

\[\begin{align*}
&\left\{ \begin{align*}
-d\nabla \left( A \left( \frac{x}{\epsilon}, Du^\epsilon \right) \right) = f \quad \text{on } \Omega, \\
&u^\epsilon \in W^{1,p}_0(\Omega).
\end{align*} \right. \\
\end{align*}\]
A good model for the study of physical behaviour of heterogeneous material is given by

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(\tau_\epsilon) \begin{cases} 
- \text{div} \left( A \left( \frac{x}{\epsilon}, Du^\epsilon \right) \right) = f & \text{on } \Omega, \\
\epsilon \in W^{1,p}_0(\Omega). 
\end{cases}
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- $\Omega$ is a bounded open set in $\mathbb{R}^n$
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- $\Omega$ is a bounded open set in $\mathbb{R}^n$
- $f$ given source term
- $\epsilon > 0$ length scale
- $u^\epsilon$ interpreted as the potential
- $A$ describes the physical properties of different materials in the body.
If $\epsilon$ is really small, a direct numerical approximation to the solution of $(\tau_\epsilon)$ may be expensive or even impossible.
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Then homogenization gives an alternative way by approximating these solutions by a function $u^H$ which solves

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The "homogenized" $b$: Physical parameters of a homogeneous body, whose behaviour is "equivalent" to the behaviour of the material with the given microstructure (Effective parameters).
\( \Omega: \) bounded open subset of \( \mathbb{R}^n \).

\( Y = (0,1)^n \): unit cube in \( \mathbb{R}^n \).
\[ \Omega: \text{bounded open subset of } \mathbb{R}^n. \]
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Let \(p \geq 2\) and \(q\) such that \(\frac{1}{p} + \frac{1}{q} = 1\).
\( \Omega \): bounded open subset of \( \mathbb{R}^n \).
\( Y = (0,1)^n \): unit cube in \( \mathbb{R}^n \).
Let \( p \geq 2 \) and \( q \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \).
\( L^p_n(D) = \left\{ u : D \to \mathbb{R}^n : \int_D |u(x)|^p \, dx < \infty \right\} \).
This space is a Banach space and its norm is defined by
\[
\|u\|_{L^p_n(D)} = \left( \int_D |u(x)|^p \, dx \right)^{1/p}.
\]
\[ W_{\text{per}}^{1,p}(Y) = \text{set of all functions } u \in W^{1,p}(Y) \text{ with mean value zero which have the same trace on the opposite faces of } Y. \]
Notation

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\[ \text{Consider } N\text{-phase materials.} \]
Notation

- \( W_{per}^{1,p}(Y) = \) set of all functions \( u \in W^{1,p}(Y) \) with mean value zero which have the same trace on the opposite faces of \( Y \).
- Consider \( N \)-phase materials.
- The characteristic function for the \( i \)-th material \( \chi_i(y) \) is \( Y \)-periodic.

\[ N \sum_{i=1}^{N} \chi_i(y) = 1. \]

A: \( \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is defined by

\[ A(y, \lambda) = \sum_{i=1}^{N} \chi_i(y) a_i |\lambda|^{p-2} \lambda, \text{ with } a_i \geq 0. \]
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\[ A(y, \lambda) = \sum_{i=1}^{N} \chi_i(y) a_i |\lambda|^{p-2} \lambda, \text{ with } a_i \geq 0. \]
For every $\lambda \in \mathbb{R}^n$, $A(\cdot, \lambda)$ is $Y$-periodic and Lebesgue measurable.
Properties of $A$

- For every $\lambda \in \mathbb{R}^n$, $A(\cdot, \lambda)$ is $Y$-periodic and Lebesgue measurable.
- Have $|A(y, 0)| = 0$ for all $y \in \mathbb{R}^n$. 
Properties of $A$

- For every $\lambda \in \mathbb{R}^n$, $A(\cdot, \lambda)$ is $Y$-periodic and Lebesgue measurable.
- Have $|A(y, 0)| = 0$ for all $y \in \mathbb{R}^n$.
- **Continuity**

\[ |A(y, \lambda_1) - A(y, \lambda_2)| \leq C_1 |\lambda_1 - \lambda_2| (|\lambda_1| + |\lambda_2| + 1)^{p-2}. \]
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**Continuity**

$$|A(y, \lambda_1) - A(y, \lambda_2)| \leq C_1 |\lambda_1 - \lambda_2| (|\lambda_1| + |\lambda_2| + 1)^{p-2}.$$

**Monotonicity**

$$(A(y, \lambda_1) - A(y, \lambda_2), \lambda_1 - \lambda_2) \geq C_2 |\lambda_1 - \lambda_2|^p$$
Set $\epsilon_k = \frac{1}{k} > 0$, $k = 1, 2, \ldots$.

$$A^{\epsilon_k}(x, \lambda) = A \left( \frac{x}{\epsilon_k}, \lambda \right)$$

$$\chi^{\epsilon_k}_i(x) = \chi_i \left( \frac{x}{\epsilon_k} \right),$$

for every $x \in \mathbb{R}^n$ and every $\lambda \in \mathbb{R}^n$. 
Set $\epsilon_k = \frac{1}{k} > 0$, $k = 1, 2, \ldots$

$$A^{\epsilon_k}(x, \lambda) = A\left(\frac{x}{\epsilon_k}, \lambda\right)$$

$$\chi_{i}^{\epsilon_k}(x) = \chi_{i}\left(\frac{x}{\epsilon_k}\right),$$

for every $x \in \mathbb{R}^n$ and every $\lambda \in \mathbb{R}^n$.

Consider the Dirichlet problem

\[
\begin{cases}
-\text{div} \left( A^{\epsilon_k}(x, \nabla u^{\epsilon_k}) \right) = f \text{ on } \Omega, \\
u^{\epsilon_k} \in W^{1,p}_0(\Omega); \\
f \in W^{-1,q}(\Omega).
\end{cases}
\]
Have $u^{\epsilon_k} \rightharpoonup u^H$ in $W^{1,p}(\Omega)$ as $\epsilon_k \to 0$, where $u^H$ is solution of

$$
\begin{aligned}
&- \text{div} \left( b \left( \nabla u^H \right) \right) = f \text{ on } \Omega, \\
&u^H \in W_0^{1,p}(\Omega);
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Preliminary Results - Homogenization Theorem

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where the monotone map $b : \mathbb{R}^n \to \mathbb{R}^n$ (independent of $f$ and $\Omega$) is defined for all $\xi \in \mathbb{R}^n$ by

$$b(\xi) = \int_Y A(y, \xi + \nabla \upsilon(y))dy,$$
Preliminary Results - Homogenization Theorem

Have \( u^{\epsilon_k} \rightharpoonup u^H \) in \( W^{1,p}(\Omega) \) as \( \epsilon_k \to 0 \), where \( u^H \) is solution of

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\[ b(\xi) = \int_Y A(y, \xi + \nabla \nu(y)) \, dy, \]

where \( \nu \) is the solution to the cell problem:

\[
\begin{cases}
\int_Y (A(y, \xi + \nabla \nu), \nabla w) \, dy = 0 \text{ for every } w \in W^{1,p}_{\text{per}}(Y), \\
\nu \in W^{1,p}_{\text{per}}(Y).
\end{cases}
\]
What do we want to do?

- In heterogeneous media the initiation of failure is a multi-scale phenomena.
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If you apply a load at the structural scale, the load is often amplified by the microstructure creating local zones of high field concentration.
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If you apply a load at the structural scale, the load is often amplified by the microstructure creating local zones of high field concentration.

Field concentrations are measured using the $L^p$ norm of the gradient field.
Dal Maso and Defranceschi: Correctors for the homogenization of Monotone Operators: There is a family of correctors depending only on $A$, which permit one to express $\nabla u^\varepsilon_k$ in terms of $\nabla u^H$ up to a rest which cvs to 0 strongly in $L^p_n$ to find lower bounds for

$$\liminf_{k \to \infty} \int_D |\nabla u^\varepsilon_k(x)|^p \, dx.$$
Y_{\varepsilon_k}^i = \varepsilon_k(i + Y), \text{ where } i \in \mathbb{Z}^n.

I_{\varepsilon_k} = \{i \in \mathbb{Z}^n : Y_{\varepsilon_k}^i \subset \Omega\}.

\Omega_k = \bigcup Y_{\varepsilon_k}^i, \text{ for } i \in I_{\varepsilon_k}.
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Let \varphi \in L^p_{\Omega}.
\( Y_{\epsilon_k}^i = \epsilon_k (i + Y) \), where \( i \in \mathbb{Z}^n \).

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\( \Omega_k = \bigcup Y_{\epsilon_k}^i \), for \( i \in I_{\epsilon_k} \).

Let \( \varphi \in L^p(\Omega) \) and \( M_{\epsilon_k} \varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a function defined by

\[
M_{\epsilon_k} (\varphi)(x) = \sum_{i \in I_{\epsilon_k}} \chi_{Y_{\epsilon_k}^i}(x) \frac{1}{|Y_{\epsilon_k}^i|} \int_{Y_{\epsilon_k}^i} \varphi(y) \, dy. \tag{2}
\]
Preliminary Results - Corrector Theory

\[ \| M_{\epsilon_k}(\varphi) - \varphi \|_{L^p(\Omega)} \to 0. \]
\begin{itemize}
  \item \( \| M_{\epsilon_k}(\varphi) - \varphi \|_{L^p(\Omega)} \to 0. \)
  \item \( M_{\epsilon_k}(\varphi) \to \varphi \) a.e. on \( \Omega. \)
\end{itemize}
Preliminary Results - Corrector Theory

- $\|M_{\epsilon_k}(\varphi) - \varphi\|_{L^p_n(\Omega)} \to 0$.
- $M_{\epsilon_k}(\varphi) \to \varphi$ a.e. on $\Omega$.
- By Jensen’s inequality: $\|M_{\epsilon_k}(\varphi)\|_{L^p_n(\Omega)} \leq \|\varphi\|_{L^p_n(\Omega)}$.
\[ \|M_{\epsilon_k}(\varphi) - \varphi\|_{L^p_n(\Omega)} \to 0. \]

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By Jensen’s inequality:

\[ \|M_{\epsilon_k}(\varphi)\|_{L^p_n(\Omega)} \leq \|\varphi\|_{L^p_n(\Omega)}. \]

\[ P : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \text{ defined by } \]

\[ P(x, \xi) = \xi + \nabla \nu(x) \]

where \( \nu \) is the unique solution of (1).
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\( P(\cdot, \xi) \) is \( Y \)-periodic and \( P_{\epsilon_k}(x, \xi) = P(\frac{x}{\epsilon_k}, \xi) \) is \( \epsilon_k \)-periodic in \( x \).
\[ \| M_{\epsilon_k}(\varphi) - \varphi \|_{L^p_n(\Omega)} \to 0. \]
\[ M_{\epsilon_k}(\varphi) \to \varphi \text{ a.e. on } \Omega. \]
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\[ P_{\epsilon_k}(\cdot, \xi) \to \xi \text{ in } L^p_n(\Omega). \]
Taking $\varphi = \nabla u^H$ in (2), we get:
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$$M_{\epsilon_k}(\nabla u^H)(x) = \sum_{i \in I_{\epsilon_k}} \chi_{Y_{\epsilon_k}^i}(x) \frac{1}{|Y_{\epsilon_k}^i|} \int_{Y_{\epsilon_k}^i} \nabla u^H(y)dy.$$
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**Corrector Theorem:**
Taking $\varphi = \nabla u^H$ in (2), we get:

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**Corrector Theorem:**

$$\left\| P \left( \frac{x}{\epsilon_k}, M_{\epsilon_k}(\nabla u^H)(x) \right) - \nabla u^\epsilon_k(x) \right\|_{L^p_n(\Omega)} \rightarrow 0,$$
By means of Young Measures and the previous Corrector Theorem, we obtain
By means of Young Measures and the previous Corrector Theorem, we obtain
\[
\int_D \int_Y \left| P(y, \nabla u^H(x)) \right|^p dydx \leq \liminf_{k \to \infty} \int_D \left| \nabla u^{\epsilon_k}(x) \right|^p dx.
\]
We want to find the same kind of bound for

\[ A(x, \lambda) = \alpha_1 \chi_1(x) |\lambda|^{\alpha_1-2} \lambda + \alpha_2 \chi_2(x) |\lambda|^{\alpha_2-2} \lambda \]

with \( \alpha_2 \geq \alpha_1 > 2 \). (Two phase Power-Law material).
Behavior of gradients of solutions to nonlinear PDEs with highly oscillatory coefficients

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$$\frac{1}{\alpha_1} + \frac{1}{\beta_2} = 1$$

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Properties of $A$

$A$ satisfies the following:

1. For all $\lambda \in \mathbb{R}^n$, $A(\cdot, \lambda)$ is $Y$-periodic and Lebesgue measurable.
2. $|A(x, 0)| = 0$ for all $x \in \mathbb{R}^n$.
3. Continuity
   
   $|A(x, \lambda_1) - A(x, \lambda_2)| \leq C_1 \chi_1(x) |\lambda_1 - \lambda_2| (1 + |\lambda_1| + |\lambda_2|)^{\alpha_1 - 2} + C'_1 \chi_2(x) |\lambda_1 - \lambda_2| (1 + |\lambda_1| + |\lambda_2|)^{\alpha_2 - 2}$

4. Monotonicity
   
   $|A(x, \lambda_1) - A(x, \lambda_2), \lambda_1 - \lambda_2| \geq C_2 \chi_1(x) |\lambda_1 - \lambda_2|^{\alpha_1} + C'_2 \chi_2(x) |\lambda_1 - \lambda_2|^{\alpha_2}$
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3. Continuity

$$|A(x, \lambda_1) - A(x, \lambda_2)|$$

$$\leq C_1 \chi_1(x) |\lambda_1 - \lambda_2| (1 + |\lambda_1| + |\lambda_2|)^{\alpha_1-2}$$

$$+ C'_1 \chi_2(x) |\lambda_1 - \lambda_2| (1 + |\lambda_1| + |\lambda_2|)^{\alpha_2-2}.$$
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$$(A(x, \lambda_1) - A(x, \lambda_2), \lambda_1 - \lambda_2) \geq C_2 \chi_1(x) |\lambda_1 - \lambda_2|^\alpha_1 + C_2' \chi_2(x) |\lambda_1 - \lambda_2|^\alpha_2.$$
Consider the Dirichlet problem

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\begin{cases}
- \text{div} \left( A^{\epsilon_k} (x, \nabla u^{\epsilon_k}) \right) = f \text{ on } \Omega, \\
u^{\epsilon_k} \in W^{1,\alpha_1}_0(\Omega); \\
f \in W^{-1,\beta_2}(\Omega).
\end{cases}
\]
Homogenization Theorem

Have $u^{\epsilon_k} \rightharpoonup u^H$ in $W^{1,\alpha_1}(\Omega)$ as $\epsilon_k \to 0$, where $u^H$ is solution of

\begin{align*}
-\text{div} \left( b(\nabla u^H) \right) &= f \quad \text{on } \Omega, \\
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where the monotone map $b: \mathbb{R}^n \to \mathbb{R}^n$ (independent of $f$ and $\Omega$) is defined for all $\xi \in \mathbb{R}^n$ by

$$b(\xi) = \int_Y A(y, \xi + \nabla \upsilon(y)) \, dy,$$

where $\upsilon$ is the solution to the cell problem:

\begin{align*}
\int_Y A(y, \xi + \nabla \upsilon(y), \nabla w(y)) \, dy &= 0 \quad \text{for every } w \in W^{1,\alpha_1}_0(Y), \\
&\upsilon \in W^{1,\alpha_1}_0(Y) \text{ per } (Y).
\end{align*}
Homogenization Theorem

Have \( u^{\epsilon_k} \rightharpoonup u^H \) in \( W^{1,\alpha_1}(\Omega) \) as \( \epsilon_k \to 0 \), where \( u^H \) is solution of

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-\text{div} \left( b \left( \nabla u^H \right) \right) = f \text{ on } \Omega, \\
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where the monotone map \( b : \mathbb{R}^n \to \mathbb{R}^n \) (independent of \( f \) and \( \Omega \)) is defined for all \( \xi \in \mathbb{R}^n \) by

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where $\nu$ is the solution to the cell problem:

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\begin{cases}
\int_Y (A(y, \xi + \nabla \nu), \nabla w) dy = 0 \text{ for every } w \in W^{1,\alpha_1}_{\text{per}}(Y), \\
\nu \in W^{1,\alpha_1}_{\text{per}}(Y).
\end{cases}
\]
What do we want to do?

- **Orlicz Norm:**

  \[
  \|f\|_{\text{Orlicz}}(\Omega) = \left[ \int_{\Omega} \chi_1(x)|f(x)|^{\alpha_1} \, dx \right]^{1/\alpha_1} + \left[ \int_{\Omega} \chi_2(x)|f(x)|^{\alpha_2} \, dx \right]^{1/\alpha_2}.
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We need a family of correctors which permit one to express \( \nabla u \in k \) in terms of \( \nabla u_{\text{H}} \) up to a rest which converges strongly in the Orlicz norm to find a lower bound as before.
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- We need a family of correctors which permit one to express \( \nabla u^{\epsilon_k} \) in terms of \( \nabla u^H \) up to a rest which cvs to 0 strongly in the **Orlicz norm** to find a lower bound as before.
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Then, if the structure is disperse, we have
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Then, if the structure is disperse, we have

$$\left\| P \left( \frac{x}{\epsilon_k}, M^{\epsilon_k} (\nabla u^H)(x) \right) - \nabla u^{\epsilon_k}(x) \right\|_{\text{Orlicz}(\Omega)} \rightarrow 0,$$

as $\epsilon_k \rightarrow 0$. 

Silvia Jiménez Bolaños

**Future work**

- **Work in progress:** Using Young Measures and the Corrector Theorem to get the lower bound for the field concentrations in the case of two phase Power-Law Material.

- What about N-Phase Power-Law Materials?
- What about the combination of a Linear with a Power-Law Material?
- What about random materials?
- What if the structure is not disperse?
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THANK YOU!


