Strong Approximation of Local Fields in Nonlinear Power Law Materials and Applications

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I would like to thank Prof. Robert Lipton for his collaboration and guidance.
Motivation - Composite Materials/Length Scales

Review of Recent Results

Mixtures of N Power Law Materials with Same Exponent

Mixtures of Two Nonlinear Power Law Materials.

Corrector Theory for Mixtures of Two Nonlinear Power Law Materials.

Application - Lower Bound on Field Concentrations.

Future work
Motivation

- Composites are materials that have inhomogeneities on length scales that are much larger than the atomic scale, but which are essentially homogeneous at macroscopic length scales.

- Fiber reinforced epoxy (Boeing 777).
In composites, failure initiation is a multiscale phenomena.

A load applied at the structural scale is often amplified by the microstructure creating local zones of high field concentration.

We study properties of local fields inside mixtures of two nonlinear power law materials.

This simple constitutive model is frequently used to describe several phenomena ranging from plasticity to optical nonlinearities in dielectric media.

This work addresses a prototypical problem in the scalar setting.
Motivation

- The goal is to develop new multiscale tools to bound the local singularity strength inside micro-structured media in terms of the macroscopic applied fields.

- The research carried out in this project draws upon the mathematical theory of Elliptic PDEs, Corrector Theory, Young Measures, and Homogenization Methods.
Review of Recent Results

Correctors for the homogenization of Monotone Operators

*Dal Maso and Defranceschi*, 1990.

Numerical homogenization and correctors for nonlinear elliptic equations


Homogenization of periodic composite power-law materials through Young measures

*Pedregal and Serrano*, 2006.

Homogenization and field concentrations in heterogeneous media

*Lipton*, 2006.
We consider $N$ nonlinear power law materials periodically distributed inside a domain $\Omega$.

$\Omega$ is an open bounded subset of $\mathbb{R}^n$, which represents a sample of the material.

The length scale of the microstructure relative to the domain is denoted by $\epsilon$.

We describe the geometry of the mixture through the characteristic functions $\chi_{i\epsilon}$, $i = 1, \ldots, N$, corresponding to each of the materials.

For $i = 1, \ldots, N$, $\chi_{i\epsilon} = 1$ in the $i$-th phase and zero outside.
Since the mixture is periodic, we use the unit period cell $Y$ to define $\chi_i^\epsilon, i = 1, \ldots, N$.

For $i = 1, \ldots, N$, the indicator function of the $i$-th phase in the unit cell $Y$ is $\chi_i(y)$.

$$\sum_{i=1}^{N} \chi_i(y) = 1.$$ 

The $\epsilon$ periodic mixture inside $\Omega$ is described by

$$\chi_i^\epsilon(x) = \chi_i(x/\epsilon) \text{ for } i = 1, \ldots, N.$$
Let $p \geq 2$ and let $q$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

The piecewise power law material is defined by the constitutive law $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$A(y, \xi) = \sum_{i=1}^{N} \chi_i(y) a_i |\xi|^{p-2} \xi, \text{ with } a_i \geq 0,$$

and the constitutive law for the $\epsilon$-periodic composite is given by

$$A_\epsilon(x, \xi) = A \left( \frac{x}{\epsilon}, \xi \right), \text{ for every } \epsilon > 0.$$
For a given source term $f \in W^{-1,q}(\Omega)$, consider

**Dirichlet Problem**

$$\begin{cases} -\text{div} (A_\epsilon (x, \nabla u_\epsilon)) = f \text{ on } \Omega, \\ u_\epsilon \in W^{1,p}_0(\Omega). \end{cases}$$

- Here $p \geq 2$ and $q$ is such that $\frac{1}{p} + \frac{1}{q} = 1$.
- $\epsilon > 0$ is length scale of the composite microstructure.
As $\epsilon \to 0$, the solutions $u_\epsilon$ of the Dirichlet problem converge weakly to $u$ in $W^{1,p}(\Omega)$, where $u$ is solution of

**Homogenized Problem**

$$
\begin{cases}
-\text{div} \left( b(\nabla u) \right) = f \text{ on } \Omega,
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
u \in W_0^{1,p}(\Omega);
\end{cases}
$$

where $b : \mathbb{R}^n \to \mathbb{R}^n$ is defined for all $\xi \in \mathbb{R}^n$ by

\[
b(\xi) = \int_Y A(y, p(y, \xi)) \, dy, \quad \text{where} \quad p(y, \xi) = \xi + \nabla \psi(y) \quad \text{solves}
\]

**Cell Problem**

\[
\begin{cases}
\int_Y (A(y, p(y, \xi)), \nabla w) \, dy = 0 \text{ for every } w \in W^{1,p}_{\text{per}}(Y),
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\psi \in W^{1,p}_{\text{per}}(Y).
\end{cases}
\]
$W_{per}^{1:p}(Y)$ denotes the set of all functions $u \in W^{1:p}(Y)$ with mean value zero which have the same trace on the opposite faces of $Y$.

As before, for $\epsilon > 0$, we rescale and define

$$p_\epsilon(x, \xi) = p \left( \frac{x}{\epsilon}, \xi \right) = \xi + \nabla u_\xi \left( \frac{x}{\epsilon} \right).$$
Mixtures of N Power Law Materials with Same Exponent - Corrector Theorem (Dal Maso and Defranceschi)

\[ Y^i_\epsilon = \epsilon(i + Y), \text{ where } i \in \mathbb{Z}^n. \]

\[ I_\epsilon = \{ i \in \mathbb{Z}^n : Y^i_\epsilon \subset \Omega \}. \]

Let \( \varphi \in L^p(\Omega, \mathbb{R}^n) \) and \( M_\epsilon \varphi : \mathbb{R}^n \to \mathbb{R}^n \) be a function defined by

\[
M_\epsilon(\varphi)(x) = \sum_{i \in I_\epsilon} \chi_{Y^i_\epsilon}(x) \frac{1}{|Y^i_\epsilon|} \int_{Y^i_\epsilon} \varphi(y) dy.
\]

\( M_\epsilon \) takes the average of the vector field in every cube.
Corrector Theorem

\[ \left\| \frac{x}{\epsilon}, M_\epsilon(\nabla u)(x) \right\| - \nabla u_\epsilon(x) \right\|_{L^p(\Omega, \mathbb{R}^n)} \to 0, \text{ as } \epsilon \to 0. \]
We use the Corrector Theorem and Young Measures to study the behavior of gradients of solutions \( \nabla u_\epsilon \) of the Dirichlet problem.

These tools allow us to bound nonlinear quantities of the gradients from below in terms of the local solution \( p \) and the homogenized gradient.
A Young measure $\nu$ is a family $\{\nu_x\}_{x \in \Omega}$ of probability measures associated with a sequence $\{f_\epsilon\}_{\epsilon > 0}$, $f_\epsilon : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that the support of $\nu_x \subset \mathbb{R}^n$ and they depend measurably on $x \in \Omega$, i.e, for all continuous $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$, the function

$$\overline{\varphi}(x) = \int_{\mathbb{R}^n} \varphi(\lambda) d\nu_x(\lambda) = \langle \varphi, \nu_x \rangle$$

is measurable.

Whenever $\varphi(f_\epsilon)$ converges weakly * in $L^\infty(\Omega)$, the weak limit is $\overline{\varphi}$, i.e,

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} \varphi(f_\epsilon) g(x) dx = \int_{\Omega} g(x) \int_{\mathbb{R}^n} \varphi(\lambda) d\nu_x(\lambda) dx$$

for all $g \in L^1(\Omega)$. 
Every bounded sequence in $L^p(\Omega; \mathbb{R}^n)$ contains a subsequence that generates a Young measure.

In order to identify the Young measure associated to a particular sequence of functions $z_\epsilon$, it is enough to check

$$\lim_{\epsilon \to 0} \int_{\Omega} \xi(x) \varphi(z_\epsilon(x)) \, dx = \int_{\Omega} \xi(x) \int_{\mathbb{R}^n} \varphi(\lambda) d\nu_x(\lambda) \, dx$$

for $\xi$ and $\varphi$ belonging to dense, countable subsets of $L^1(\Omega)$ and $C_0(\mathbb{R}^n)$, respectively.
Let \( \{z_\varepsilon\} \) and \( \{w_\varepsilon\} \) bounded sequences in \( L^p(\Omega; \mathbb{R}^n) \). If

\[
\|w_\varepsilon - z_\varepsilon\|_{L^p(\Omega; \mathbb{R}^n)} \to 0, \text{ as } \varepsilon \to 0,
\]

then \( \{z_\varepsilon\} \) and \( \{w_\varepsilon\} \) share the same Young measure.

If \( \{z_\varepsilon\} \) is a sequence of measurable functions with associated Young measure \( \nu = \{\nu_x\}_{x \in \Omega} \), then for all Caratheodory function \( \psi \geq 0 \) and \( D \subset \Omega \) measurable we have

\[
\int_D \int_{\mathbb{R}^n} \psi(x, \lambda) d\nu_x(\lambda) dx \leq \liminf_{\varepsilon \to 0} \int_D \psi(x, z_\varepsilon(x)) dx.
\]
By Corrector Theorem, we have \( \left\{ p \left( \frac{x}{\epsilon}, M_\epsilon(\nabla u)(x) \right) \right\} \) and \( \{ \nabla u_\epsilon(x) \} \) have the same Young Measure, i.e. we have \( \nu = \{ \nu_x \}_{x \in \Omega} \) such that

\[
\int_\Omega \zeta(x) \int_{\mathbb{R}^n} \phi(\lambda) d\nu_x(\lambda) dx \leq \lim_{\epsilon \to 0} \int_\Omega \zeta(x) \phi \left( p \left( \frac{x}{\epsilon}, M_\epsilon(\nabla u)(x) \right) \right) dx
\]

\[
\int_\Omega \zeta(x) \int_{\mathbb{R}^n} \phi(\lambda) d\nu_x(\lambda) dx \leq \lim_{\epsilon \to 0} \int_\Omega \zeta(x) \phi \left( \nabla u_\epsilon(x) \right) dx
\]

For all \( \phi \in C_0(\mathbb{R}^n) \) and for all \( \zeta \in C_0^\infty(\mathbb{R}^n) \).
We prove
\[
\lim_{\epsilon \to 0} \int_{\Omega} \zeta(x) \phi \left( p \left( \frac{x}{\epsilon}, M_\epsilon (\nabla u)(x) \right) \right) \, dx
= \int_{\Omega} \zeta(x) \int_{Y} \phi(p(y, \nabla u(x))) \, dy \, dx.
\]

Consequence:
\[
\int_{\Omega} \zeta(x) \int_{\mathbb{R}^n} \phi(\lambda) \, d\nu_x(\lambda) \, dx
= \int_{\Omega} \zeta(x) \int_{Y} \phi(p(y, \nabla u(x))) \, dy \, dx.
\]

for all \( \phi \in C_0(\mathbb{R}^n) \) and for all \( \zeta \in C_0^\infty(\mathbb{R}^n) \).
Mixtures of N Power Law Materials with Same Exponent - Lower bound on Field Concentrations

We have that for all Caratheodory function $\psi \geq 0$ and $E \subset \Omega$ measurable

**Lower bound**

$$\int_D \int_Y \psi(x, (p(y, \nabla u(x)))) dy dx \leq \liminf_{\epsilon \to 0} \int_D \psi(x, \nabla u_\epsilon(x)) dx.$$ 

In particular, for $r > 1$

$$\int_D \int_Y |p(y, \nabla u(x))|^r dy dx \leq \liminf_{\epsilon \to 0} \int_D |\nabla u_\epsilon(x)|^r dx.$$

- If the sequence $\psi(x, \nabla u_\epsilon(x))$ is weakly convergent in $L^1(\Omega)$, then the inequality becomes an equality.
We consider two nonlinear power law materials periodically distributed inside a domain $\Omega$.

$\Omega$ is an open bounded subset of $\mathbb{R}^n$, which represents a sample of the material.

The length scale of the microstructure relative to the domain is denoted by $\epsilon$.

We describe the geometry of the mixture through the characteristic functions $\chi_1^\epsilon$ and $\chi_2^\epsilon$ corresponding to each of the materials.

$\chi_1^\epsilon = 1$ in material one and zero outside, and $\chi_2^\epsilon = 1 - \chi_1^\epsilon$. 
Since the mixture is periodic, we use the unit period cell $Y$ to define $\chi_1^\epsilon$ and $\chi_2^\epsilon$.

The indicator function of phase one in the unit cell $Y$ is $\chi_1(y)$ and the indicator functions of phase two is $\chi_2(y) = 1 - \chi_1(y)$.

The $\epsilon$ periodic mixture inside $\Omega$ is described by

$$\chi_1^\epsilon(x) = \chi_1(x/\epsilon), \text{ and } \chi_2^\epsilon(x) = \chi_2(x/\epsilon).$$
The exponents for each of the materials are denoted $\alpha_1$ and $\alpha_2$ and satisfy $2 \leq \alpha_1 \leq \alpha_2$, and we denote their Hölder conjugates by $\beta_2$ and $\beta_1$ respectively.

We assume that the microstructure is made of inclusions of the material corresponding to the exponent $\alpha_1$ surrounded by the material corresponding to the exponent $\alpha_2$. This class of microstructure is often referred as disperse microstructure.

The piecewise power law material is defined by the constitutive law $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$A(x, \xi) = \alpha_1 \chi_1(x) |\xi|^{\alpha_1-2} \xi + \alpha_2 \chi_2(x) |\xi|^{\alpha_2-2} \xi,$$

and the constitutive law for the $\epsilon$-periodic composite is given by

$$A_\epsilon(x, \xi) = A\left(\frac{x}{\epsilon}, \xi\right), \text{ for every } \epsilon > 0.$$
Behavior of gradients of solutions to nonlinear PDEs with highly oscillatory coefficients

For a given source term \( f \in W^{-1,\beta_2}(\Omega) \), consider

**Dirichlet Problem**

\[
\begin{aligned}
-\text{div} \left( A_\epsilon (x, \nabla u_\epsilon) \right) &= f \text{ on } \Omega, \\
u_\epsilon &\in W^{1,\alpha_1}_0(\Omega).
\end{aligned}
\]

- \( 2 \leq \alpha_1 \leq \alpha_2 \) with \( \frac{1}{\alpha_1} + \frac{1}{\beta_2} = 1 \) and \( \frac{1}{\alpha_2} + \frac{1}{\beta_1} = 1 \).
- \( \epsilon > 0 \) is length scale of the composite microstructure.
- A priori estimates give:

\[
\int_{\Omega} \chi_1^\epsilon |\nabla u_\epsilon(x)|^{\alpha_1} dx + \int_{\Omega} \chi_2^\epsilon |\nabla u_\epsilon(x)|^{\alpha_2} dx < C, \text{ for all } \epsilon > 0.
\]
As $\epsilon \to 0$, the solutions $u_\epsilon$ of the Dirichlet problem converge weakly to $u$ in $W^{1,\alpha_1}(\Omega)$, where $u$ is solution of

**Homogenized Problem**

\[
\begin{cases}
-\text{div} (b (\nabla u)) = f \text{ on } \Omega, \\
u \in W^{1,\alpha_1}_0(\Omega);
\end{cases}
\]

where $b : \mathbb{R}^n \to \mathbb{R}^n$ is defined for all $\xi \in \mathbb{R}^n$ by

\[b(\xi) = \int_Y A(y, p(y, \xi)) dy,\]

where $p(y, \xi) = \xi + \nabla \upsilon(y)$ solves

**Cell Problem**

\[
\begin{cases}
\int_Y (A(y, \xi + \nabla \upsilon), \nabla w) dy = 0 \text{ for every } w \in W^{1,\alpha_1}_{\text{per}}(Y), \\
\upsilon \in W^{1,\alpha_1}_{\text{per}}(Y).
\end{cases}
\]
Mixtures of Two Nonlinear Power Law Materials - Properties of $b$

- **Continuity**

$$|b(\xi_1) - b(\xi_2)| \leq K_1 \left( |\xi_1 - \xi_2|^{\frac{1}{\alpha_1 - 1}} + |\xi_1 - \xi_2|^{\frac{1}{\alpha_2 - 1}} \right).$$

- **Monotonicity**

$$(b(\xi_1) - b(\xi_2), \xi_1 - \xi_2) \geq K_2 \left[ \int_Y \chi_1(y) |p(y, \xi_1) - p(y, \xi_2)|^{\alpha_1} dy 
+ \int_Y \chi_2(y) |p(y, \xi_1) - p(y, \xi_2)|^{\alpha_2} dy \right] \geq 0.$$ 

where $K_1, K_2$ are strictly positive constants.
$W_{\text{per}}^{1,\alpha_i}(Y) \ (i = 1, 2)$ denotes the set of all functions $u \in W^{1,\alpha_i}(Y) \ (i = 1, 2)$ with mean value zero which have the same trace on the opposite faces of $Y$.

As before, for $\epsilon > 0$, we rescale and define

$$p_\epsilon(x, \xi) = p \left( \frac{x}{\epsilon}, \xi \right) = \xi + \nabla u_\xi \left( \frac{x}{\epsilon} \right).$$
The homogenization theorem allows us to approximate $\nabla u_\epsilon$ in terms of $\nabla u$ up to a remainder which converges to 0 weakly in $L^{\alpha_1}(\Omega; \mathbb{R}^n)$.

We develop new strong convergence results that capture the asymptotic behavior of the gradients $\nabla u_\epsilon$, as $\epsilon$ tends to 0. Here we focus on the gradients inside each phase $\chi_i^\epsilon \nabla u_\epsilon$, $i = 1, 2$, and develop strong approximations for these quantities.
Mixtures of Two Nonlinear Power Law Materials - Corrector Theorem

- $Y^i_\epsilon = \epsilon(i + Y)$, where $i \in \mathbb{Z}^n$.
- $I_\epsilon = \{ i \in \mathbb{Z}^n : Y^i_\epsilon \subset \Omega \}$.

Let $\varphi \in L^{\alpha_2}(\Omega, \mathbb{R}^n)$ and $M_\epsilon \varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a function defined by

$$M_\epsilon(\varphi)(x) = \sum_{i \in I_\epsilon} \chi_{Y^i_\epsilon}(x) \frac{1}{|Y^i_\epsilon|} \int_{Y^i_\epsilon} \varphi(y) dy.$$ 

$M_\epsilon$ takes the average of the vector field in every cube.

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We get the following result:

\[
\int_{\Omega} \left| \chi_i^\epsilon(x) p_\epsilon(x, M_\epsilon(\nabla u)(x)) - \chi_i^\epsilon(x) \nabla u_\epsilon(x) \right|^{\alpha_i} \, dx \to 0,
\]

as \( \epsilon \to 0 \) for \( i = 1, 2 \).
For $i = 1, 2$, we have

\[
\int_D \int_Y \Psi(x, \chi_i(y)p(y, \nabla u(x))) \, dy \, dx \\
\leq \liminf_{\epsilon \to 0} \int_D \Psi(x, \chi_{i}^{\epsilon}(x) \nabla u_{\epsilon}(x)) \, dx,
\]

for $D \subset \Omega$ measurable, and for all continuous functions $\Psi \geq 0$.

- Functions of the form $\Psi$ are often used in failure criteria (Tsai-Hahn/Tsai-Hill/Tsai-Wu).
- If the sequence $\Psi(x, \nabla u_{\epsilon}(x))$ is weakly convergent in $L^1(\Omega)$, then the inequality becomes an equality.
Failure Criteria

For $i = 1, 2$, let $x$ in $i$-th material.

Suppose you know that $\Psi(x, \chi_i^\epsilon(x) \nabla u_\epsilon(x)) < F_i$ implies no damage initiated, and

$\Psi(x, \chi_i^\epsilon(x) \nabla u_\epsilon(x)) > F_i$ on set of non zero measure means that damage started.

Note that if $\int_Y \Psi(x, \chi_i(y)p(y, \nabla u(x))) \, dy > F_i$, then damage is initiated.
In particular,

\[ \int_{D} \int_{Y} \chi_i(y) |p(y, \nabla u(x))|^r \, dy \, dx = \int_{D} \int_{Y} |\chi_i(y)p(y, \nabla u(x))|^r \, dy \, dx \]

\[ \leq \liminf_{\epsilon \to 0} \int_{D} |\chi_i^\epsilon(x) \nabla u_\epsilon(x)|^p \, dx = \int_{D} \chi_i^\epsilon(x) |\nabla u_\epsilon(x)|^p \, dx, \]

for any \( r > 1 \).
Future work

- What about random materials?
- What if the structure is not disperse?
- Study the case of Laminates: Even though they are not disperse maybe the same results can be obtained.


