Math 112 Fall 2006 Exam 3 Solutions

Some solutions include the details; others have just the final answer.

1. (a) $\lim\limits_{x\to 0^+} \dfrac{1}{e^x-1} - \dfrac{1}{x} = \lim\limits_{x\to 0^+} \dfrac{x-e^x+1}{x(e^x-1)} \overset{\text{L'H}}{=} \lim\limits_{x\to 0^+} \dfrac{1-e^x}{xe^x+e^x-1} \overset{\text{L'H}}{=} \lim\limits_{x\to 0^+} \dfrac{-e^x}{(x+2)e^x} = -\dfrac{1}{2}$

   (b) $\lim\limits_{x\to\infty} \left(\dfrac{x-2}{x}\right)^x$ has the indefinite form "1^∞". Let $y = \left(\frac{x-2}{x}\right)^x$, so $\ln y = x\ln\left(\frac{x-2}{x}\right)$. Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x\ln\left(\frac{x-2}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{x-2}{x}\right)}{\frac{1}{x}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\left(\frac{x}{x-2}\right)\left(\frac{2}{x^2}\right)}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{-2x}{x-2} = -2$$

   Then

$$\lim_{x\to\infty} \left(\frac{x-2}{x}\right)^x = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{-2}$$

   (c) $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{1+\cos\theta} = 0$

2. $\displaystyle\int_2^\infty \frac{dx}{(x-7)^4} = \lim_{t\to 7^-} \int_2^t \frac{dx}{(x-7)^4} + \lim_{t\to 7^+} \int_t^8 \frac{dx}{(x-7)^4} + \lim_{t\to\infty} \int_8^t \frac{dx}{(x-7)^4}$

   and

$$\lim_{t\to 7^-} \int_2^t \frac{dx}{(x-7)^4} = \lim_{t\to 7^-} \left(-\frac{1}{3}(x-7)^{-3}\Big|_2^t\right) = \lim_{t\to 7^-} \left(-\frac{1}{3(t-7)^3} + \frac{1}{3(2-7)^3}\right) = \infty$$

   That is, the limit is not a finite number, so the improper integral is *divergent*.

3. $\displaystyle\int_4^\infty \frac{dx}{x\sqrt{x-1}} = \lim_{t\to\infty} \int_4^t \frac{dx}{x\sqrt{x-1}}.$

   Let $u = \sqrt{x-1} = (x-1)^{1/2}$. Then $du = \frac{1}{2}(x-1)^{-1/2}$, $u^2+1 = x$, and $2u\,du = dx$. So

$$\int_4^t \frac{dx}{x\sqrt{x-1}} = \int_{\sqrt{3}}^{\sqrt{t-1}} \frac{2}{u^2+1}\,du = 2(\arctan(\sqrt{t-1}) - \arctan\sqrt{3}) = 2\left(\arctan(\sqrt{t-1}) - \frac{\pi}{3}\right)$$

   So

$$\int_4^\infty \frac{dx}{x\sqrt{x-1}} = \lim_{t\to\infty} 2\left(\arctan(\sqrt{t-1}) - \frac{\pi}{3}\right) = 2\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{\pi}{3}$$

4. We know

$$-1 \le \sin(2x) \le 1$$

   so

$$-2 \le 2\sin(2x) \le 2 \quad \text{and therefore} \quad 0 \le 2\sin(2x)+2 \le 4.$$

   (We really only need the result that $2\sin(2x)+2 \ge 0$.) Then

$$3x^2 + 2\sin(2x) + 2 \ge 3x^2$$

   so

$$0 \le \frac{1}{3x^2+2\sin(2x)+2} \le \frac{1}{3x^2}$$

   Since $\displaystyle\int_1^\infty \frac{dx}{3x^2} = \frac{1}{3}\int_1^\infty \frac{dx}{x^2}$ is convergent, by the comparison theorem we conclude that

$$\int_1^\infty \frac{1}{3x^2+2\sin(2x)+2}\,dx$$

   is *convergent*.

5. $(0,0)$ and $(1/2,1/2)$.

6. The area is $A = 2\left(\int_0^{\pi/6} \frac{1}{2}(\sin\theta)^2\,d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{2}(1-\sin\theta)^2\,d\theta\right)$

7. First, we find that the curve is at $(5,3)$ when $t = 0$.

   We will need $\dfrac{dx}{dt} = 15t^4 + 4$ and $\dfrac{dy}{dt} = 6t + 2$.

   Then
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t+2}{15t^4+4},$$
   so at $t = 0$,
$$\frac{dy}{dx} = \frac{1}{2}.$$
   For $\frac{d^2y}{dx^2}$, we will need
$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{6t+2}{15t^4+4}\right) = \frac{(15t^4+4)(6) - (6t+2)(60t^3)}{(15t^4+4)^2}$$
   Then
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{(15t^4+4)(6) - (6t+2)(60t^3)}{(15t^4+4)^3}$$
   and at $t = 0$, we find
$$\frac{d^2y}{dx^2} = \frac{3}{8}$$

8. (a) $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} \overset{\text{L'H}}{=} \lim\limits_{\theta\to 0} \dfrac{\cos\theta}{1} = 1$ so we must choose $A = 1$.

   (b) *(The sketch will be shown in class.)*

9. (a) From the relation $\tan\theta = \dfrac{y}{x}$, we conclude $\dfrac{y}{x} = 5$, or $y = 5x$. This is a *straight line through the origin with slope 5*.

   (b) We will use the $r^2 = x^2 + y^2$ and $x = r\cos\theta$ in the following:
$$\begin{aligned} r &= \frac{1}{1+\cos\theta} \\ r + r\cos\theta &= 1 \\ r &= 1 - r\cos\theta \\ r^2 &= (1 - r\cos\theta)^2 \\ x^2 + y^2 &= (1-x)^2 = 1 - 2x + x^2 \\ y^2 &= 1 - 2x \end{aligned}$$
   This last equation shows that the curve is a *parabola*.

10. $\displaystyle\int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_0^{2\pi} \sqrt{\left(e^{\theta/2}\right)^2 + \left(\frac{e^{\theta/2}}{2}\right)^2}\,d\theta = \int_0^{2\pi} \sqrt{\frac{5}{4}e^{\theta}}\,d\theta = \frac{\sqrt{5}}{2}\int_0^{2\pi} e^{\theta/2} = \sqrt{5}e^{\theta/2}\Big|_0^{2\pi}$
    $= \sqrt{5}(e^{\pi} - 1)$