Math 113 $\boxed{\text{Exam 3 Solutions}}$ Fall '05

1. Let

$$f(x,y) = e^{x^2+xy-y}$$

(a) Find a vector that is perpendicular to the contour line of f at the point $(0,2)$.

(b) Find a vector that is perpendicular to the surface $z = f(x,y)$ at the point $(0,2,e^{-2})$.

(c) Find the equation of the plane that is tangent to the surface $z = f(x,y)$ at the point $(0,2,e^{-2})$.

(d) Find the rate of change of f (with respect to distance) in the direction $\vec{i}+\vec{j}$ at the point $(0,2)$.

(e) Find the degree 2 Taylor polynomial that approximates $f(x,y)$ near $(0,2)$.

(a) $\operatorname{grad} f(x,y) = (2x+y)e^{x^2+xy-y}\vec{i} + (x-1)e^{x^2+xy-y}\vec{j}$

$\operatorname{grad} f(0,2) = \boxed{2e^{-2}\vec{i} - e^{-2}\vec{j}}$

(b) $\boxed{2e^{-2}\vec{i} - e^{-2}\vec{j} - \vec{k}}$

(c) $\boxed{2e^{-2}x - e^{-2}(y-2) - (z-e^{-2}) = 0}$ OR $\boxed{z = e^{-2} + 2e^{-2}x - e^{-2}(y-2)}$

(d) $\vec{u} = \frac{1}{\sqrt{2}}(\vec{i}+\vec{j})$ $\quad f_{\vec{u}}(0,2) = \operatorname{grad} f(0,2)\cdot\vec{u} = \frac{2e^{-2}-e^{-2}}{\sqrt{2}} = \boxed{\frac{e^{-2}}{\sqrt{2}}}$

(e) $f_{xx}(x,y) = (2x+y)^2e^{x^2+xy-y} + 2e^{x^2+xy-y}$ $\quad f_{xx}(0,2) = 6e^{-2}$

$f_{xy}(x,y) = (2x+y)(x-1)e^{x^2+xy-y} + e^{x^2+xy-y}$ $\quad f_{xy}(0,2) = -e^{-2}$

$f_{yy}(x,y) = (x-1)^2e^{x^2+xy-y}$ $\quad f_{yy}(0,2) = e^{-2}$

$\boxed{Q(x,y) = e^{-2} + 2e^{-2}x - e^{-2}(y-2) + 3e^{-2}x^2 - e^{-2}x(y-2) + \frac{1}{2}e^{-2}(y-2)^2}$

2. Suppose a surface is defined implicitly by the equation

$$z^3 + e^{xz} - xy + y^3 = 1,$$

and the path of particle is confined to remain on the surface. When particle is at the point $(1, 1, 0)$, the x coordinate of the particle is increasing at the rate of 2 cm/sec, and the y coordinate is decreasing at the rate of 1 cm/sec. Find the rate of change of the z coordinate of the particle at this instant.

Let $f(x,y,z) = z^3 + e^{xz} - xy + y^3$, so the surface is given by

$f(x,y,z) = 1.$

Differentiate with respect to t, and use the chain rule:

(★) $f_x(x,y,z)\frac{dx}{dt} + f_y(x,y,z)\frac{dy}{dt} + f_z(x,y,z)\frac{dz}{dt} = 0$

We were given $\frac{dx}{dt} = 2$, and $\frac{dy}{dt} = -1$.

Also

$f_x(x,y,z) = ze^{xz} - y$ $\qquad f_x(1,1,0) = -1$

$f_y(x,y,z) = -x + 3y^2$ $\qquad f_y(1,1,0) = 2$

$f_z(x,y,z) = 3z^2 + xe^{xz}$ $\qquad f_z(1,1,0) = 1$

So (★) becomes

$(-1)(2) + (2)(-1) + (1)\frac{dz}{dt} = 0 \Rightarrow \boxed{\frac{dz}{dt} = 4 \text{ cm/sec}}$

OR

grad $f(1,1,0) = -\vec{i} + 2\vec{j} + \vec{k}$.

The velocity vector of the particle is $\vec{v} = 2\vec{i} - \vec{j} + \frac{dz}{dt}\vec{k}$

Since the particle remains on the surface, the velocity vector must be tangent to the surface, which means $\vec{v}$ must be perpendicular to grad $f(1,1,0)$. That is, grad $f(1,1,0) \cdot \vec{v} = 0$.

$\Rightarrow -2 - 2 + \frac{dz}{dt} = 0 \Rightarrow \boxed{\frac{dz}{dt} = 4 \text{ cm/sec}}$

3. The path of a particle is given by

$$x = 3 - t, \quad y = 2 + \frac{1}{t}, \quad z = t + t^2.$$

(a) Find the parametric equations of the line that is tangent to the path at the point $(2, 3, 2)$.

(b) What is the magnitude of the acceleration of the particle when it is at the point $(2, 3, 2)$?

(a) First note that the particle is at $(2,3,2)$ when $\underline{t=1}$.

The vector form of the path of the particle is

$$\vec{r}(t) = (3-t)\vec{i} + \left(2+\frac{1}{t}\right)\vec{j} + (t+t^2)\vec{k}.$$

Then $$\vec{r}\,'(t) = -\vec{i} - \frac{1}{t^2}\vec{j} + (1+2t)\vec{k}$$

and $$\vec{r}\,'(1) = -\vec{i} - \vec{j} + 3\vec{k}.$$

$\vec{r}\,'(1)$ gives the velocity vector at $t=1$; this vector gives the direction of the tangent line.

The tangent line is

$$\boxed{\vec{r} = \vec{r}_0 + t\vec{v}, \quad \text{where } \vec{r}_0 = 2\vec{i} + 3\vec{j} + 2\vec{k}, \ \vec{v} = \vec{r}\,'(1) = -\vec{i} - \vec{j} + 3\vec{k}}$$

or

$$\boxed{x = 2-t, \quad y = 3-t, \quad z = 2+3t}$$

(b) The acceleration is

$$\vec{r}\,''(t) = \frac{2}{t^3}\vec{j} + 2\vec{k}$$

$$\vec{r}\,''(1) = 2\vec{j} + 2\vec{k}$$

$$\boxed{\|\vec{r}\,''(1)\| = \sqrt{2^2+2^2} = \sqrt{8}}$$

4. *An assortment of unrelated short problems...*

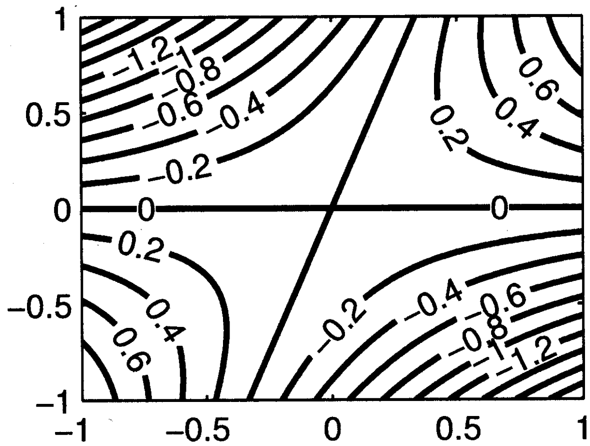
(a) Suppose $f(x, y)$ satisfies $f(0,0) = 0$, $f_x(0,0) =$ and $f_y(0,0) = 0$. Match each of the following cases to one of the contour plots.

i. $f_{xx}(0,0) < 0$, $f_{xy}(0,0) > 0$, $f_{yy}(0,0) = 0$ Contour Plot C

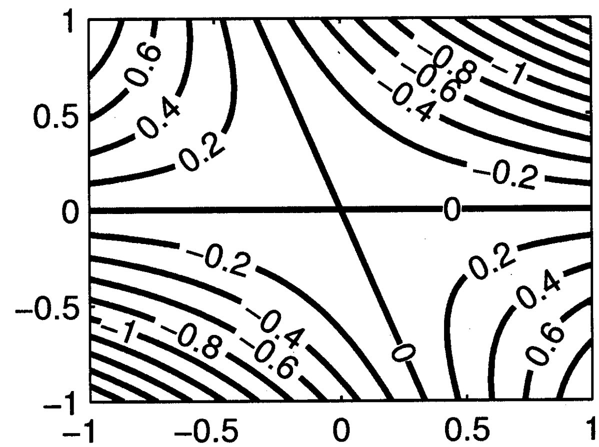
ii. $f_{xx}(0,0) < 0$, $f_{xy}(0,0) < 0$, $f_{yy}(0,0) = 0$ Contour Plot D

iii. $f_{xx}(0,0) = 0$, $f_{xy}(0,0) < 0$, $f_{yy}(0,0) < 0$ Contour Plot B

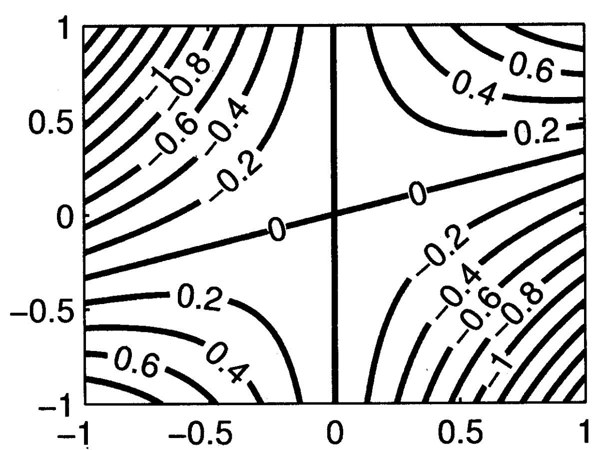
A



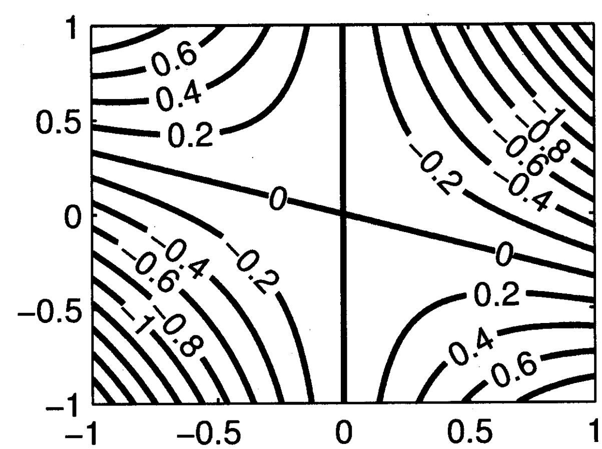
B



C



D



(b) Let

$$f(x, y) = \sin(2x) + x^2 - xy + y.$$

$\boxed{\text{Yes.}}$

At $(0, 0)$, are there any directions in which the rate of change of f (with respect to distance) is exactly 1? Clearly explain your answer. (There will be no partial credit for simply saying "yes" or "no", even if you give the correct answer.)

$\text{grad } f(x,y) = (2\cos(2x) + 2x - y)\vec{i} + (-x+1)\vec{j}$

$\text{grad } f(0,0) = 2\vec{i} + \vec{j}$

$\|\text{grad } f(0,0)\| = \sqrt{5} > 1$

$f_{\vec{u}}(0,0) = \text{grad } f(0,0) \cdot \vec{u} = \|\text{grad } f(0,0)\| \cos\theta$

$= \sqrt{5}\cos\theta$ where θ is the angle between $\vec{u}$ and $\text{grad } f(0,0)$.

As θ is varied, we have $-1 \le \cos\theta \le 1$, so $-\sqrt{5} \le f_{\vec{u}}(0,0) \le \sqrt{5}$, so there are directions (two, in fact) where $f_{\vec{u}}(0,0) = 1$

OR Let $\vec{u} = u_1\vec{i} + u_2\vec{j}$ be a unit vector, so $u_1^2 + u_2^2 = 1$.

We want $f_{\vec{u}}(0,0) = 1$

$\text{grad } f(0,0) \cdot \vec{u} = 1$

$2u_1 + u_2 = 1$

$u_2 = 1 - 2u_1$

$\Rightarrow u_1^2 + (1-2u_1)^2 = 1$

$5u_1^2 - 4u_1 = 0$

$u_1 = 0$ or $u_1 = \frac{4}{5}$

$\Rightarrow \boxed{\vec{u} = \vec{j} \quad \text{OR} \quad \vec{u} = -\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}}$

(c) Let

$$f(x, y) = \sqrt{(x - y - 2)^2 + x^2}$$

Where is this function not differentiable?

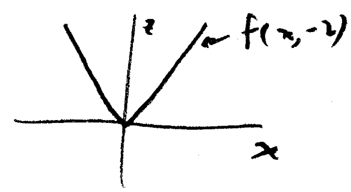
The only possible points are where

$(x - y - 2)^2 + x^2 = 0$

$\Rightarrow x = 0, y = -2$ ie the point $(0, -2)$.

In the cross section $y = -2$ we see

$f(x,-2) = \sqrt{2x^2} = \sqrt{2}|x|$



The graph has "corner" at $(0,-2)$, so

$\boxed{f \text{ is not differentiable at } (0,-2)}$

(d) Suppose all derivatives of the function $f(x, y)$ are continuous and

$$f_{xy}(x,y) = x \quad \text{for all } (x,y).$$

Indicate whether each of the following statements is ***true*** or ***false***. Justify your answers.

i. $f_{yx}(x,y) = y$

FALSE. Since all derivatives of f are continuous, we know

$$f_{yx}(x,y) = f_{xy}(x,y) = x.$$

ii. $f_{xx}(x,y) = y$

FALSE. $f_{xy}(x,y) = x$ for all (x,y) implies $f_x(x,y) = xy + g(x)$ for some function $g(x)$. Then

$$f_{xx}(x,y) = y + g'(x)$$

Example: $f(x,y) = \frac{1}{2}x^2 y + x^2$

$$f_{xy}(x,y) = x \quad \text{but} \quad f_{xx}(x,y) = y + 2$$

iii. $f_{yy}(x,y)$ does not depend on x.

TRUE. $f_{xy}(x,y) = x$ for all x implies $f_y(x,y) = \frac{1}{2}x^2 + h(y)$ for some function $h(y)$. Then

$$f_{yy}(x,y) = h'(y),$$

which does not depend on x.

5. In a first calculus course, you learn that a line tangent to the graph of $f(x)$ at $x = x_0$ may be written

$$y = f(x_0) + f'(x_0)(x - x_0). \tag{1}$$

(a) Consider a function of two variables. The notation $f(\vec{\mathbf{x}})$ has the same meaning as $f(x, y)$ where $\vec{\mathbf{x}} = x\vec{\imath} + y\vec{\jmath}$. Show that the equation of the plane tangent to the graph of f at the point (x_0, y_0) may be written

$$z = f(\vec{\mathbf{x}}_0) + \text{grad} f(\vec{\mathbf{x}}_0) \bullet (\vec{\mathbf{x}} - \vec{\mathbf{x}}_0) \tag{2}$$

where $\vec{\mathbf{x}}_0 = x_0\vec{\imath} + y_0\vec{\jmath}$ and $\bullet$ indicates the dot product.

The tangent plane is $z = f(\vec{x}_0) + f_x(\vec{x}_0)(x-x_0) + f_y(\vec{x}_0)(y-y_0)$

Now, $\text{grad}\, f(\vec{x}_0) = f_x(\vec{x}_0)\vec{i} + f_y(\vec{x}_0)\vec{j}$, and $\vec{x} - \vec{x}_0 = (x-x_0)\vec{i} + (y-y_0)\vec{j}$

so $\boxed{\text{grad} f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = f_x(\vec{x}_0)(x-x_0) + f_y(\vec{x}_0)(y-y_0)}$

and therefore (2) is the equation of the tangent plane,

(b) Consider the vector function $\vec{\mathbf{f}}(t)$, where $\vec{\mathbf{f}}$ is now a vector in (x, y, z) space. That is, $\vec{\mathbf{r}} = \vec{\mathbf{f}}(t)$ is a parameterized curve in space, where $\mathbf{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$. Derive a formula for the line that is tangent to the curve at $t = t_0$ that has the same structure as (1) and (2). Your answer should be something like "The tangent line is $\vec{\mathbf{r}} = ...$"; fill in the dots.

The tangent line at $t = t_0$ is

$$\boxed{\vec{r} = \vec{f}(t_0) + \vec{f}'(t_0)(t - t_0)}$$

(c) The previous three formulas (that is, (1), (2) and your answer to (b)) are all examples of Taylor polynomials of degree 1. For the vector function $\vec{\mathbf{f}}(t)$ of part (b), what is the formula for the degree 2 Taylor polynomial at $t = t_0$? Briefly explain your answer.

$$\boxed{\vec{r} = \vec{f}(t_0) + \vec{f}'(t_0)(t - t_0) + \frac{\vec{f}''(t_0)}{2}(t - t_0)^2}$$

At $t = t_0$, this curve has the same position, velocity and acceleration as $\vec{f}(t)$.