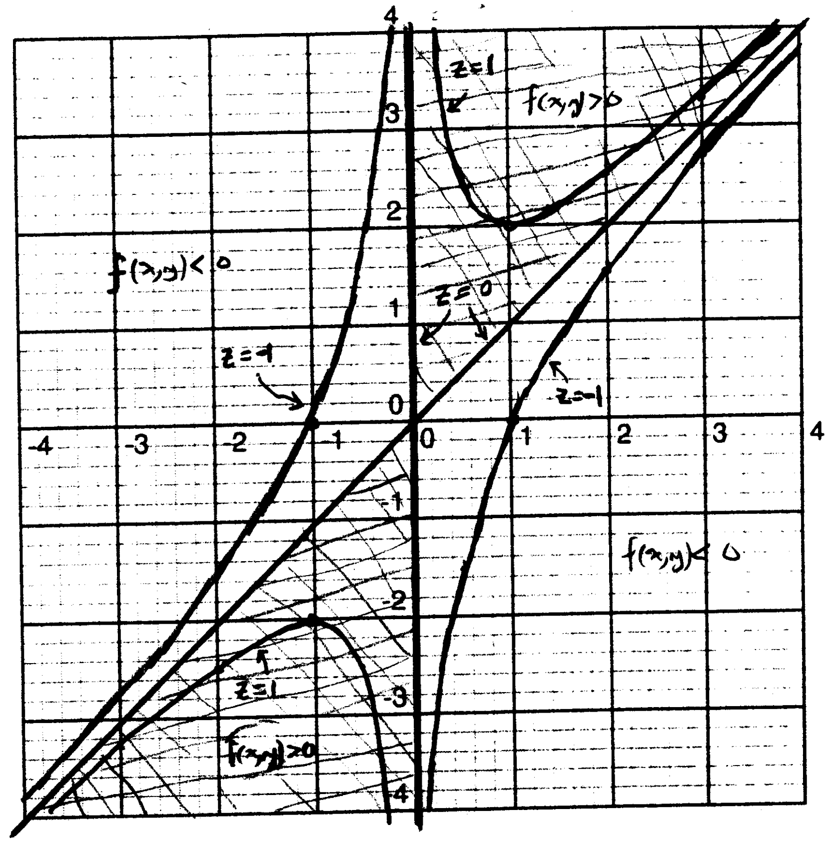
Math 113, Fall '05 Exam 1 Solutions

1. Let $f(x, y) = x(y - x)$.

(a) In the following axes, sketch the contour lines for the contour values 0, 1, and -1. *Clearly label each line with its contour value.*



$x(y-x) = 0 \Rightarrow x = 0$ OR $y = x$

$x(y-x) = 1 \Rightarrow y = x + \frac{1}{x}$

$x(y-x) = -1 \Rightarrow y = x - \frac{1}{x}$

Shading (cross-hatching) indicates the region where $f(x,y) > 0$.

(b) In the above plot, indicate the regions where $f(x, y) > 0$, and the regions where $f(x, y) < 0$.

(c) Describe the shape of the graph of f.

The graph is saddle-shaped.
(It is a hyperbolic paraboloid.)

2. For each of the following functions, describe in words the level set $g(x, y, z) = 1$, *and* determine if it can be expressed as the graph of a function $f(x, y)$. If it can be expressed as a graph, give the function $f(x, y)$. If it can not, explain why not.

(a) $g(x, y, z) = \ln(x^2 + y^2 + 2z^2)$

$\ln(x^2+y^2+2z^2)=1 \Rightarrow x^2+y^2+2z^2=e$ This is an ellipsoid.

This set can not be expressed as the graph of a function $f(x,y)$. If we try to solve for z, we have

$$z = \pm\frac{1}{2}\sqrt{e-x^2-y^2},$$

which is two functions. (That is, the set fails the "vertical line test.")

(b) $g(x, y, z) = \cos(3x - 2y - z)$

$\cos(3x-2y-z)=1 \Rightarrow 3x-2y-z=2\pi k$, where k is an integer.

This set is an infinite set of parallel planes. Each integer k gives a plane.

Because there are infinitely many planes, this set can not be expressed as the graph of a function $f(x,y)$.

(c) $g(x, y, z) = \dfrac{z}{x^2 + y^2 + 1}$

$\dfrac{z}{x^2+y^2+1} = 1 \Rightarrow z = x^2 + y^2 + 1$

This is an elliptical paraboloid, opening upwards, with the smallest value on the z axis at (0,0,1).

This level set is the graph of $\boxed{f(x,y) = x^2 + y^2 + 1}$

3. Find the following.

(1) (2) (3)

(a) The equation of a plane that contains the points (1, 1, 1), (1, 2, 3), and (2, 1, 6).

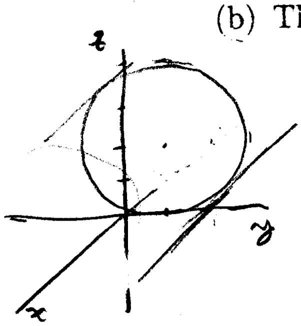
$$m = \frac{\Delta z}{\Delta x}\Big|_{y \text{ constant}} = \frac{z_3 - z_1}{x_3 - x_1} = \frac{6-1}{2-1} = 5 \qquad n = \frac{\Delta z}{\Delta y}\Big|_{x \text{ constant}} = \frac{z_2 - z_1}{y_2 - y_1} = \frac{3-1}{2-1} = 2$$

so $z = mx + ny + c = 5x + 2y + c$

At $x=1$, $y=1$, we want $z=1$, so $1 = 7 + c \Rightarrow c = -6$

$$\Rightarrow \boxed{z = 5x + 2y - 6}$$

(b) The equation of a circular cylinder with radius 2 around the line $y = 1$, $z = 2$.



$$(y-1)^2 + (z-2)^2 = 4$$

This is a circle with radius 2 in the (y, z) plane, so in (x, y, z) space, it is a cylinder around the line $y=1$ and $z=2$.

(c) A function $g(x, y, z)$ whose level surfaces are spheres centered at (3, 2, 0).

$$g(x,y,z) = (x-3)^2 + (y-2)^2 + z^2$$

(other answers are possible)

4. Some questions on limits and continuity.

(a) Let $f(x, y) = \dfrac{x^2 y}{x^3 + y^3}$. Show that $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

On the x axis ($y=0$), we have

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{0}{x^3} = 0$$

but on the line $y=x$, we have

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^3}{2x^3} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

Because these two paths result in different limits at $(0,0)$, we know that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(b) Let $f(x, y) = \dfrac{\sin\left(\sqrt{x^2 + y^2}\right)}{\sqrt{x^2 + y^2}}$.

Determine if $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ exists. If so, find the limit. If not, explain why not.

Let $r = \sqrt{x^2+y^2}$. Then $r \to 0$ if and only if $(x,y)\to(0,0)$.

so

$$\lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{\sin r}{r} \overset{\text{L'H}}{=} \lim_{r\to 0} \frac{\cos r}{1} = \cos 0 = 1$$

(c) Let

$$f(x,y) = \begin{cases} \frac{5x\sin(x) + y(x+5)(e^y - 1)}{x^2+y^2} & (x,y) \neq (0,0) \\ k & (x,y) = (0,0) \end{cases}$$

Find the value of k for which f is continuous at $(0,0)$. (You may assume that there is such a value; you do not have to *prove* continuity.) Briefly explain how you determined k.

f is continuous at $(0,0)$ if $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.

Since $f(0,0) = k$, we require

$$k = \lim_{(x,y)\to(0,0)} f(x,y).$$

We are told that the limit exists, so to find it, we can use any convenient path to create a single variable limit, and then evaluate it.

On the x axis, where $y = 0$, we have

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{5x\sin x}{x^2} = 5 \lim_{x\to 0} \frac{\sin x}{x} = 5$$

so to make f continuous at $(0,0)$, we choose $\boxed{k=5}$.