14.1/ 3.

(a) Negative. If the price of beef, b, increases, we generally expect people to buy less beef. So if b increases, we expect Q to decrease, which means $\partial Q/\partial b < 0$.

(b) Positive. If the price of chicken increases, while the price of beef remains constant, people will tend to buy more beef. So if c increases, Q increases, thus $\partial Q/\partial c > 0$.

(c) The rate of change of the quantity of beef purchased with respect to the price of the beef is -213 (kg/week)/(dollar/kg). Roughly speaking, since

$$\frac{\Delta Q}{\Delta b} \approx \frac{-213}{1},$$

an increase in the price of beef by one dollar per kilogram would decrease the quantity of beef sold by 213 kilogram per week.

14.1/ 5.

(a) The units of $\frac{\partial c}{\partial x}$ are concentration/distance. For example, if the concentration is measured in mg/ℓ and the distance is measured in centimeters, then the units of $\frac{\partial c}{\partial x}$ are (mg/ℓ)/cm.

$\frac{\partial c}{\partial x}$ is the rate of change of the concentration with respect to distance. It gives how fast the concentration changes with respect to x at a fixed time t. (Often $\frac{\partial c}{\partial x}$ is referred to as the *concentration gradient.*)

Since the concentration will be highest at the point of injection and decrease at x increases, we expect $\frac{\partial c}{\partial x} < 0$, at least initially. (Whether or not $\frac{\partial c}{\partial x}$ could become positive depends on precisely how the drug concentration changes as the blood carries it along in the blood vessel. If the drug remains localized in the blood for some time, but the blood is moving down the blood vessel, there there will be a maximum in the concentration that will move down the blood vessel. After this maximum passes a fixed point x, $\frac{\partial c}{\partial x}$ will be positive at that point.)

(b) The units of $\frac{\partial c}{\partial t}$ are concentration/distance. For example, if the concentration is measured in mg/ℓ and the time is measured in minutes, then the units of $\frac{\partial c}{\partial t}$ are (mg/ℓ)/min.

$\frac{\partial c}{\partial t}$ is the rate of change of the concentration with respect to time at a fixed point x and time t. At a given point x downstream from the injection point, we expect that at first, the concentration will increase. As the drug passes by (and possible spreads out), the concentration will reach a maximum and then decrease. Therefore, at first we expect $\frac{\partial c}{\partial t} > 0$, and later $\frac{\partial c}{\partial t} < 0$.

14.1/ 17.

(a) (i) $f_x(A) > 0$. If you *increase* the x coordinate from point A, the value of the function *increases.*

(ii) $f_y(A) < 0$. If you *increase* the y coordinate from point A, the value of the function *decreases.*

(b) As the point P moves along a straight line from A to B, $f_x(P)$ changes from positive to negative, while $f_y(P)$ remains negative.

14.1/20

(a) Since $f_x > 0$, the values on the contours increase as you move to the right. Since $f_y > 0$, the values on the contours increase as you move upward. See Figure 14.1.

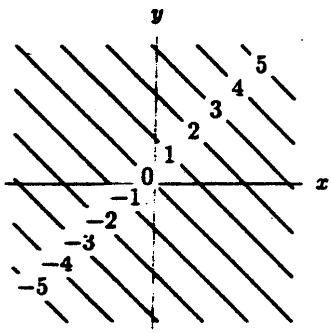


Figure 14.1: $f_x > 0$ and $f_y > 0$

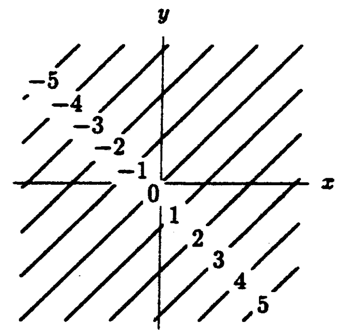


Figure 14.2: $f_x > 0$ and $f_y < 0$

(b) Since $f_x > 0$, the values on the contours increase as you move to the right. Since $f_y < 0$, the values on the contours decrease as you move upward. See Figure 14.2.

(c) Since $f_x < 0$, the values on the contours decrease as you move to the right. Since $f_y > 0$, the values on the contours increase as you move upward. See Figure 14.3.

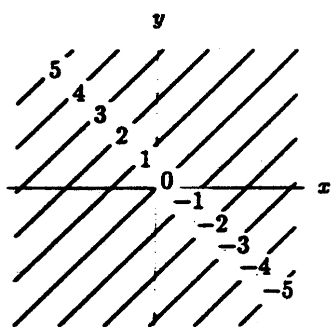


Figure 14.3: $f_x < 0$ and $f_y > 0$

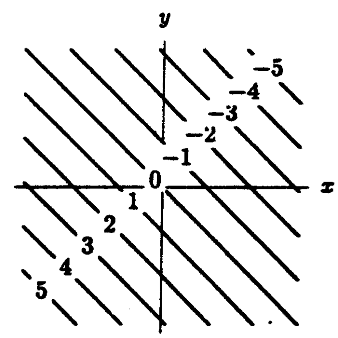


Figure 14.4: $f_x < 0$ and $f_y < 0$

(d) Since $f_x < 0$, the values on the contours decrease as you move to the right. Since $f_y < 0$, the values on the contours decrease as you move upward. See Figure 14.4.

(Note: A few problems are out of order.)

$\boxed{14.2/4}$ $z = (x^2+x-y)^7$

$$\frac{\partial z}{\partial x} = 7(x^2+x-y)^6(2x+1)$$

$$\frac{\partial z}{\partial y} = 7(x^2+x-y)^6(-1) = -7(x^2+x-y)^6$$

$\boxed{14.2/6}$ $V = \frac{1}{3}\pi r^2 h$ $\quad V_r = \frac{2}{3}\pi r h$

$\boxed{14.2/10}$

$$\frac{\partial}{\partial x}\left(xe^{\sqrt{xy}}\right) = \frac{\partial}{\partial x}\left(xe^{(xy)^{\frac{1}{2}}}\right)$$

Product Rule

$$= x\frac{\partial}{\partial x}\left(e^{(xy)^{\frac{1}{2}}}\right) + \left(\frac{\partial}{\partial x}(x)\right)e^{(xy)^{\frac{1}{2}}}$$

chain Rule

$$= xe^{(xy)^{\frac{1}{2}}}\left(\tfrac{1}{2}(xy)^{-\frac{1}{2}}(y)\right) + e^{(xy)^{\frac{1}{2}}}$$

$$= \frac{1}{2}\sqrt{xy}\,e^{\sqrt{xy}} + e^{\sqrt{xy}}$$

$$= e^{\sqrt{xy}}\left(\frac{1}{2}\sqrt{xy}+1\right)$$

$\boxed{14.2/21}$ $\frac{\partial}{\partial t}\left(v_0 t + \frac{1}{2}at^2\right) = v_0 + at$

$\boxed{14.2/31}$ $Q = c(a_1K^{b_1} + a_2L^{b_2})^{\gamma}$

$$\frac{\partial Q}{\partial K} = \gamma c(a_1K^{b_1}+a_2L^{b_2})^{\gamma-1}(a_1b_1K^{b_1-1})$$

$$= \gamma a_1 b_1 c K^{b_1-1}(a_1K^{b_1}+a_2L^{b_2})^{\gamma-1}$$

$\boxed{14.2/7}$

(NOT ASSIGNED)

$$z = \frac{3x^2y^7 - y^2}{15xy - 8}$$

$$z_y = \frac{(15xy-8)(21x^2y^6 - 2y) - (3x^2y^7 - y^2)(15x)}{(15xy-8)^2}$$

$$= \frac{270x^3y^7 - 168x^2y^6 - 15xy^2 + 16y}{(15xy-8)^2}$$

$\boxed{14.2/9}$

$$\frac{\partial}{\partial x}(a\sqrt{x}) = \frac{\partial}{\partial x}(ax^{\frac{1}{2}}) = \frac{a}{2}x^{-\frac{1}{2}} = \frac{a}{2\sqrt{x}}$$

$\boxed{14.2/22}$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi}(LC)^{-\frac{1}{2}}$$

$$\frac{\partial f_0}{\partial L} = \left(\frac{1}{2\pi}\right)\left(-\frac{1}{2}\right)(LC)^{-\frac{3}{2}}(C)$$

$$= -\frac{1}{4\pi}\frac{C}{(LC)^{3/2}}$$

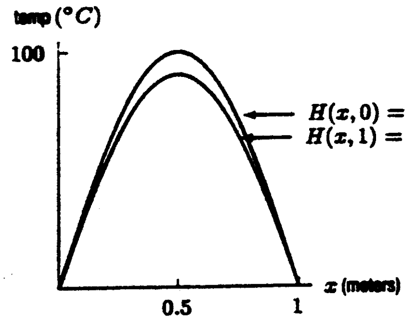
$$= \frac{-1}{4\pi L\sqrt{LC}}$$

14.2/39

(a) Substituting $t = 0$ and $t = 1$ into the formula for H gives:

$$H(x,0) = 100\sin(\pi x)$$
$$H(x,1) = 100e^{-0.1}\sin(\pi x) = 90.5\sin(\pi x).$$

The graphs of $H(x,0)$ and $H(x,1)$ are shown below.



(b) To calculate $H_x(x,t)$, we hold t constant and differentiate with respect to x:

$$H_x(x,t) = \frac{\partial}{\partial x}H(x,t) = \frac{\partial}{\partial x}(100e^{-0.1t}\sin(\pi x)) = 100\pi e^{-0.1t}\cos(\pi x)$$
$$H_x(0.2,t) = 100\pi e^{-0.1t}\cos(0.2\pi) = 254.2e^{-0.1t}\ ^\circ\text{C/meter}$$
$$H_x(0.8,t) = 100\pi e^{-0.1t}\cos(0.8\pi) = -254.2e^{-0.1t}\ ^\circ\text{C/meter.}$$

The practical interpretation of these partial derivatives is the rate of change in temperature at $x = 0.2$ and $x = 0.8$ as we increase the distance from the end $x = 0$. Notice that $e^{-0.1t}$ is positive for all t. Given the formula for $H(x,t)$, we see that the closer the position to the center of the rod, the hotter the temperature. The partial derivative $H_x(0.2,t)$ has a positive sign because, at $x = 0.2$ as we increase x, we get closer to the center of the rod which is hottest. The partial derivative $H_x(0.8,t)$ has a negative sign because, at $x = 0.8$ as we increase x, we get further away from the center of the rod which is hottest.

(c) To calculate $H_t(x,t)$, we hold x constant and differentiate with respect to t:

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial t}(100e^{-0.1t}\sin(\pi x)) = -10e^{-0.1t}\sin(\pi x)\ ^\circ\text{C/second.}$$

For all t, and for $0 < x < 1$ (that is, for all t and all x inside the rod), the partial derivative $H_t(x,t)$ is negative. In terms of heat, $H_t(x,t)$ represents the rate at which the temperature of the rod is changing as time passes at position x and time t. Thus, the temperature inside the rod is always decreasing.

$\boxed{14.2/45}$

$$f_x(x,y) = 4x^3y^2 - 3y^4$$

Integrate with respect to x (holding y constant) to get

$$f(x,y) = x^4y^2 - 3xy^4 + C(y)$$

Note that instead of an arbitrary constant, we have an <u>arbitrary function of y, $C(y)$</u> (You can check that this is correct by taking the partial derivative with respect to x.)

Now differentiate with respect to y to obtain

$$f_y(x,y) = 2x^4y - 12xy^3 + C'(y)$$

In order for this to match the expression given in the problem, we must have $C'(y) = 0$. Thus $C(y) = C_0$, where <u>C_0 is an arbitrary constant.</u>

So any function of the form

$$\boxed{f(x,y) = x^4y^2 - 3xy^4 + C_0}$$

has the given partial derivatives.

14.3/ 3. To find the tangent plane, we use the formula given on page 652 (and in class):

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

In this case, $f(x,y) = e^y + x + x^2 + 6$, $a = 1$, $b = 0$, and $f(1,0) = e^0 + 1 + 1^2 + 6 = 9$. We compute

$$f_x(x,y) = 1 + 2x, \quad \text{and} \quad f_y(x,y) = e^y,$$

so

$$f_x(1,0) = 3, \quad \text{and} \quad f_y(1,0) = 1.$$

The equation of the tangent plane is then

$$z = 9 + (3)(x-1) + (1)(y-0),$$

or

$$z = 6 + 3x + y.$$

14.3/ 6. We have $z = f(x,y) = \sin(xy)$. The differential dz (or df) is

$$\begin{aligned} dz &= f_x(x,y)dx + f_y(x,y)dy \\ &= y\cos(xy)dx + x\cos(xy)dy. \end{aligned}$$

14.3/ 9. $df = e^{-y}dx - xe^{-y}dy$, so at the point $(1,0)$, $df = dx - dy$.

14.3/ 13. The *local linearization* is just the tangent plane, interpreted as an approximation to f near the point (a,b). Thus, the formula is

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

In this case, $f(x,y) = x^2y$, $a = 3$ and $b = 1$, so $f(a,b) = f(3,1) = 9$. We find

$$f_x(x,y) = 2xy, \quad \text{and} \quad f_y(x,y) = x^2,$$

so $f_x(3,1) = 6$ and $f_y(3,1) = 9$. Thus,

$$f(x,y) \approx 9 + 6(x-3) + 9(y-1) = 6x + 9y - 18.$$

14.3/ 14.

(a) The answer given must be incorrect because it is not the equation of a plane. The equation of a plane is a linear function of the form $z = mx + ny + c$, where m, n, and c are *constants.*

(b) The student was trying to use the formula

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b),$$

with $f(x,y) = x^3 - y^2$, $a = 2$ and $b = 3$. So the student computed $f(2,3) = 8 - 9 = 1$, and also computed

$$f_x(x,y) = 3x^2, \quad f_y(x,y) = -2y.$$

At this point the student failed to *first evaluate the partial derivatives at* (a,b), and therefore obtained the incorrect answer given in the book.

(c) The student should have computed

$$f_x(2,3) = 12, \quad \text{and} \quad f_y(2,3) = -6,$$

to obtain the correct answer

$$z = 12(x-2) - 6(y-3) - 1,$$

or

$$z = 12x - 6y - 7.$$

14.3/ 18. We use the tangent plane as an approximation to $T(x,y)$ near $(2,1)$:

$$\begin{aligned} T(x,y) &\approx T(2,1) + T_x(2,1)(x-2) + T_y(2,1)(y-1) \\ &= 135 + 16(x-2) - 15(y-1). \end{aligned}$$

Then

$$T(2.04, 0.97) \approx 135 + 16(2.04-2) - 15(0.97-1) = 135 + (16)(0.04) + (15)(0.03) = 136.09.$$

$\boxed{14.3/22}$

(a) When $V = 25$ and $P = 1$, we have $T = 304.9$. The differential dT is

$$dT = \frac{\partial T}{\partial V}\,dV + \frac{\partial T}{\partial P}\,dP = (-16.574\frac{1}{V^2} + 1.06\frac{1}{V^3} + 12.187P)\,dV + (-0.3879 + 12.187V)\,dP.$$

When $V = 25$ and $P = 1$ this is

$$dT = 12.16\,dV + 304.29\,dP.$$

(b) If $\Delta P = 0.1$ and $\Delta T = 0$, then

$$0 \approx (12.16)\Delta V + (304.29)(0.1),$$

so

$$\Delta V \approx -\frac{30.429}{12.16} \approx -2.5.$$

Thus the volume would have to decrease by about 2.5 dm^3, or about 10%.