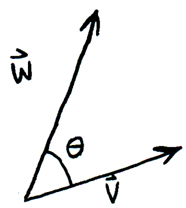
Math 113 Fall '05

Homework 7 Solutions

$\boxed{13.3/14}$



$\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$

(a) If $\|\vec{v}\|$ increases, then $\vec{v}\cdot\vec{w}$ <u>increases</u>.

(b) If θ increases, then $\cos\theta$ decreases, so $\vec{v}\cdot\vec{w}$ <u>decreases</u>.

$\boxed{13.3/18}$

Two vectors $\vec{u}$ and $\vec{v}$ are perpendicular if $\vec{u}\cdot\vec{v} = 0$.

$\vec{u}\cdot\vec{v} = t^2 - t - 2$.

Solve $t^2 - t - 2 = 0 \Rightarrow t = \dfrac{1 \pm \sqrt{1-(-8)}}{2} = \dfrac{1\pm 3}{2} = -\dfrac{2}{2}, \dfrac{4}{2}$

$= -1, 2$.

so $\vec{u}$ and $\vec{v}$ are perpendicular if $\boxed{t = -1 \text{ or } t = 2}$.

The vectors are parallel if one is a multiple of the other; that is, if $\vec{u} = c\vec{v}$ for some constant c.

This means

$$t\vec{i} - \vec{j} + \vec{k} = ct\vec{i} + ct\vec{j} - 2c\vec{k},$$

and so we require

$$t = ct, \ \underline{\text{and}} \quad -1 = ct, \ \underline{\text{and}} \quad 1 = -2c.$$

The last equality gives $c = -\frac{1}{2}$, and then the first gives $t = 0$, but then the second says $-1 = 0$.

So there is <u>no solution</u>, and thus there is no value of t for which the vectors are parallel.

$\boxed{13.3/22}$ $\vec{n} = -\vec{i} + 2\vec{j} + \vec{k}$, $\Rightarrow$ $a = -1$, $b = 2$, $c = 1$

$(x_0, y_0, z_0) = (1, 0, 2)$

The plane is given by

$$(-1)(x-1) + (2)(y-0) + (1)(z-2) = 0$$

or $$-x + 2y + z = 1$$

(NOT ASSIGNED)

$\boxed{13.3/24}$ $\vec{n} = 2\vec{i} - 3\vec{j} + 7\vec{k}$ $\Rightarrow$ $a = 2$, $b = -3$, $c = 7$

$(x_0, y_0, z_0) = (1, -1, 2)$

The plane is given by

$$(2)(x-1) + (-3)(y+1) + (7)(z-2) = 0$$

or $$2x - 3y + 7z = 19$$

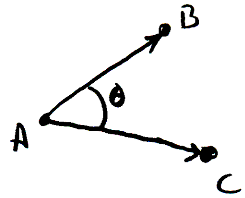
$\boxed{13.3/26}$ $\vec{n} = 3\vec{i} + \vec{j} + \vec{k}$ is normal to the given plane, so it is also normal to any plane parallel to the given plane. So $a = 3$, $b = 1$, $c = 1$, and $(x_0, y_0, z_0) = (-2, 3, 2)$.

The plane is given by

$$(3)(x+2) + (1)(y-3) + (1)(z-2) = 0$$

or $$3x + y + z = -1$$

$\boxed{13.3/30}$



$$\vec{AB} = (4-2)\vec{i} + (2-2)\vec{j} + (1-2)\vec{k} = 2\vec{i} - \vec{k}.$$

$$\vec{AC} = (2-2)\vec{i} + (3-2)\vec{j} + (1-2)\vec{k} = \vec{j} - \vec{k}$$

Then

$$\cos\theta = \frac{\vec{AB}\cdot\vec{AC}}{\|\vec{AB}\|\,\|\vec{AC}\|} = \frac{1}{\sqrt{5}\,\sqrt{2}} = \frac{1}{\sqrt{10}}$$

so

$$\theta = \arccos\left(\frac{1}{\sqrt{10}}\right) \approx 1.249 \text{ radians} \quad (71.57 \text{ degrees})$$

13.3/33 We need to find the speed of the wind in the direction of the track. Looking at Figure 13.23, we see that we want the component of $\vec{w}$ in the direction of $\vec{v}$. We calculate

$$\begin{aligned}\|\vec{w}_{\text{parallel}}\| = \|\vec{w}\|\cos\theta = \frac{\vec{w}\cdot\vec{v}}{\|\vec{v}\|} &= \frac{(5\vec{i}+\vec{j})\cdot(2\vec{i}+6\vec{j})}{\|2\vec{i}+6\vec{j}\|}\\ &= \frac{16}{\sqrt{40}} \approx 2.53\\ &< 5\end{aligned}$$

Therefore, the race results will not be disqualified.

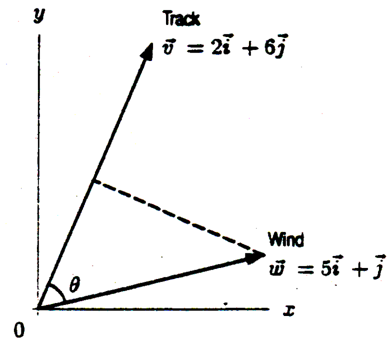


Figure 13.23

13.3/38 **(a)** The geometric definition of the dot product says that

$$\vec{n}\cdot\overrightarrow{P_0P} = \|\vec{n}\|\|\overrightarrow{P_0P}\|\cos\theta,$$

where θ is the angle between $\vec{n}$ and $\overrightarrow{P_0P}$ with $0 \le \theta \le \pi$. To say that the dot product $\vec{n}\cdot\overrightarrow{P_0P}$ is positive means that the angle between $\vec{n}$ and $\overrightarrow{P_0P}$ is between 0 and $\pi/2$, and strictly less than $\pi/2$. Hence $\vec{n}$ and $\overrightarrow{P_0P}$ are both pointing to the same side of the plane. Thus, all the points satisfying $\vec{n}\cdot\overrightarrow{P_0P} > 0$ are on the same side of the plane, the side which $\vec{n}$ points to. To say that the dot product is negative is to say that $\pi/2 < \theta \le \pi$, and this means that $\overrightarrow{P_0P}$ and $\vec{n}$ are pointing to opposite sides of the plane. Thus, all points satisfying $\vec{n}\cdot\overrightarrow{P_0P} < 0$ are on the side of the plane opposite to $\vec{n}$.

(b) Suppose the normal vector is $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$, let $P_0 = (x_0, y_0, z_0)$ be a point in the plane and let $P = (x, y, z)$ be a variable point. Then $\overrightarrow{P_0P} = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}$. Then $\vec{n}\cdot\overrightarrow{P_0P} > 0$ means

$$a(x - x_0) + b(y - y_0) + c(z - z_0) > 0$$

and $\vec{n}\cdot\overrightarrow{P_0P} < 0$ means

$$a(x - x_0) + b(y - y_0) + c(z - z_0) < 0$$

If the equation of the plane is written $ax + by + cz = d$ (with $d = ax_0 + by_0 + cz_0$) then the inequalities become

$$ax + by + cz > d \quad\text{and}\quad ax + by + cz < d.$$

(c) We test each of the points $P = (-1, -1, 1)$, $Q = (-1, -1, -1)$ and $R = (1, 1, 1)$, using the coordinate version of the inequalites in part (b):

$$\begin{aligned}P:&\quad 2\cdot(-1) - 3\cdot(-1) + 4\cdot 1 = 5 > 4\\ Q:&\quad 2\cdot(-1) - 3\cdot(-1) + 4\cdot(-1) = -3 < 4\\ R:&\quad 2\cdot 1 - 3\cdot 1 + 4\cdot 1 = 3 < 4\end{aligned}$$

Therefore Q and R are on the same side of the plane as each other; P is on the other side.

13.3/41 They are perpendicular if their dot product is 0.

$$\begin{aligned}\left[(\vec{b}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{c})\vec{b}\right]\cdot\vec{c} &= ((\vec{b}\cdot\vec{c})\vec{a})\cdot\vec{c} - ((\vec{a}\cdot\vec{c})b)\cdot\vec{c}\\ &= (\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{c}) - (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{c})\\ &= 0\end{aligned}$$

$\boxed{13.4/4}$ $\vec{v} = \vec{i} + \vec{j} + \vec{k}$, $\vec{w} = \vec{i} + \vec{j} - \vec{k}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \boxed{-2\vec{i} + 2\vec{j}}$$

$\boxed{13.4/7}$ $((\vec{i} + \vec{j}) \times \vec{i}) \times \vec{j} = (\vec{i} \times \vec{i} + \vec{j} \times \vec{i}) \times \vec{j}$

$$= (\vec{0} - \vec{k}) \times \vec{j}$$

$$= -\vec{k} \times \vec{j}$$

$$= \vec{i}$$

$\boxed{13.4/18}$

(a) A vector normal to the plane is

$$\vec{n} = \vec{AB} \times \vec{AC} = (4\vec{i} - \vec{j} + 4\vec{k}) \times (2\vec{i} - 4\vec{j} + 5\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & 4 \\ 2 & -4 & 5 \end{vmatrix} = 11\vec{i} - 12\vec{j} - 14\vec{k}$$

Using A for the point in the plane, we have the equation

$$(11)(x+1) + (-12)(y-3) + (-14)z = 0$$

or $\boxed{11x - 12y - 14z = -47}$

(b) The area of the parallelogram formed by $\vec{AB}$ and $\vec{AC}$ is $\|\vec{AB} \times \vec{AC}\| = \sqrt{11^2 + (-12)^2 + (-14)^2} = \sqrt{461}$.

So the area of the triangle formed by the three points is $\boxed{\dfrac{\sqrt{461}}{2}}$

$\boxed{13.4/20}$

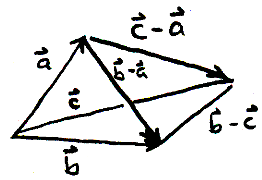
Normal vectors for the two planes are

$$\vec{n}_1 = 4\vec{i} - 3\vec{j} + 2\vec{k} \quad \text{and} \quad \vec{n}_2 = \vec{i} + 5\vec{j} - \vec{k}$$

The line of intersection is perpendicular to both these vectors, so a vector parallel to the line of intersection is

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -3 & 2 \\ 1 & 5 & -1 \end{vmatrix} = \boxed{-7\vec{i} + 6\vec{j} + 23\vec{k}}$$

$\boxed{13.4/34}$



The four outward pointing area vectors are $\frac{1}{2}(\vec{b}\times\vec{a})$, $\frac{1}{2}(\vec{a}\times\vec{c})$, $\frac{1}{2}(\vec{c}\times\vec{b})$, and

and $\frac{1}{2}(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})$, and

$$\frac{1}{2}(\vec{b}\times\vec{a}) + \frac{1}{2}(\vec{a}\times\vec{c}) + \frac{1}{2}(\vec{c}\times\vec{b}) + \frac{1}{2}(\vec{b}-\vec{a})\times(\vec{c}-\vec{a}).$$

$$= \frac{1}{2}\left[\cancel{\vec{b}\times\vec{a}} + \cancel{\vec{a}\times\vec{c}} + \vec{c}\times\vec{b} + (\vec{b}\times\vec{c} - \cancel{\vec{a}\times\vec{c}} - \cancel{\vec{b}\times\vec{a}} + \underbrace{\vec{a}\times\vec{a}}_{0})\right]$$

$$= \frac{1}{2}\left[\vec{c}\times\vec{b} - (\vec{c}\times\vec{b})\right]$$

$$= \vec{0}.$$