Math 113, Fall '05 Homework 9 Solutions

$\boxed{14.4/18}$

$$z = f(x,y) = xe^y$$

$$\text{grad } f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j} = \boxed{e^y\vec{i} + xe^y\vec{j}}$$

$\boxed{14.4/27}$

$$f(\alpha,\beta) = \frac{2\alpha+3\beta}{2\alpha-3\beta}$$

$$\text{grad } f(\alpha,\beta) = f_\alpha(\alpha,\beta)\vec{i} + f_\beta(\alpha,\beta)\vec{j}$$

$$= \frac{(2\alpha-3\beta)(2)-(2\alpha+3\beta)(2)}{(2\alpha-3\beta)^2}\vec{i} + \frac{(2\alpha-3\beta)(3)-(2\alpha+3\beta)(-3)}{(2\alpha-3\beta)^2}\vec{j}$$

$$= \frac{-12\beta}{(2\alpha-3\beta)^2}\vec{i} + \frac{12\alpha}{(2\alpha-3\beta)^2}\vec{j}$$

$\boxed{14.4/33}$

$$f(x,y) = (x^2+y^2)^{-1} \qquad f_x(x,y) = \frac{-2x}{(x^2+y^2)^2}, \quad f_y(x,y) = \frac{-2y}{(x^2+y^2)^2}$$

so

$$\text{grad } f(-1,3) = f_x(1,3)\vec{i} + f_y(1,3)\vec{j}$$

$$= \frac{2}{100}\vec{i} + \frac{-6}{100}\vec{j}$$

$$= \frac{1}{50}(\vec{i} - 3\vec{j})$$

$\boxed{14.4/37}$ $f(x,y) = xy + y^3$. $\quad f_x(x,y) = y \quad f_y(x,y) = x + 3y^2$

$$f_{\vec{u}}(1,2) = \text{grad} f(1,2)\cdot\vec{u} = \left(f_x(1,2)\vec{i} + f_y(1,2)\vec{j}\right)\cdot\left(\tfrac{3}{5}\vec{i} - \tfrac{4}{5}\vec{j}\right)$$

$$= (2\vec{i} + 13\vec{j})\left(\tfrac{3}{5}\vec{i} - \tfrac{4}{5}\vec{j}\right)$$

$$= \frac{6}{5} - \frac{52}{5} = -\frac{46}{5}$$

(Note that $\vec{u}$ is a unit vector, so we can use it as is.)

14.4/45

The gradient vectors are perpendicular to the level curves. To determine the length of the gradient vector, we estimate the rate of change of the function from the contour diagram. At $(1,1)$, the value of f changes from 1 to 2 in a distance of $\sqrt{2}$ (as it moves from $(1,1)$ to $(2,2)$), so the length of $\mathbf{grad}\, f$ is $\frac{1}{\sqrt{2}} \approx 0.7$. At $(1,4)$, the value of f changes slightly faster (the lines are closer together), so $\mathbf{grad}\, f$ is slightly longer here. In fact, the value of f changes from 2 to 3 as we move from $(1,4)$ to $(2.2, 4.2)$ a distance of about 1.2. So the new length is $1/1.2 = 0.8$.

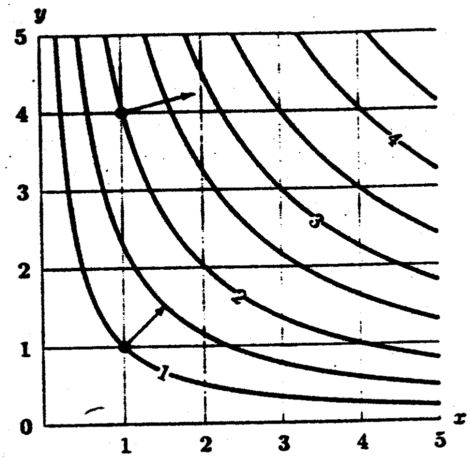


Figure 14.9: Gradient Vectors

$\boxed{14.4/55}$

$f(x,y) = x^2y^3$, $\quad \text{grad} f(x,y) = 2xy^3\vec{i} + 3x^2y^2\vec{j}$

(a) The gradient vector points in the direction of the maximum rate of change:

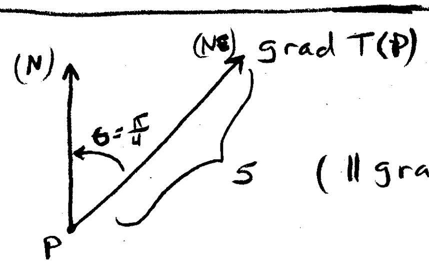
$$\text{grad} f(-1,2) = -16\vec{i} + 12\vec{j}$$

(b) $-\text{grad} f(-1,2) = 16\vec{i} - 12\vec{j}$ points in the direction of minimum rate of change.

(c) The vector $12\vec{i} + 16\vec{j}$ is <u>perpendicular to $\text{grad} f(-1,2)$</u>, so it points in a direction in which the rate of change is zero (i.e. tangent to the level curve of f through $(-1,2)$).

NOTE-THIS PROBLEM IS OUT OF ORDER!

$\boxed{14.4/67}$



(where T is the temp.)

$(\|\text{grad}\, T(P)\| = 5)$

We want $T_{\vec{u}}(P)$, where $\vec{u} = \vec{k}$. We have

$$T_{\vec{u}}(P) = \|\text{grad}\, T(P)\| \cos\theta = 5\cos\left(\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{2}$$

$\boxed{14.4/60}$

The equation of the curve is $x^2+y^2=13$, and at $(2,3)$, we have $2^2+3^2=4+9=13$, so $(2,3)$ is on the curve.

Let $f(x,y)=x^2+y^2$. Then the curve is the contour line of f, given by $f(x,y)=13$.

Now $\quad \text{grad}\, f(x,y)=2x\vec{i}+2y\vec{j}$,

and $\text{grad}\, f(2,3)=\boxed{4\vec{i}+6\vec{j}}$. This vector is normal to the curve at $(2,3)$. The tangent line is then

$$4(x-2)+6(y-3)=0$$

or

$$4x+6y=26$$

14.4/64 → NEXT PAGE

$\boxed{14.4/66}$ (a) The graph of T has circular symmetry, so the level curves of T are circles.

(b) The denominator is smallest when $x^2+y^2=0$, so T is greatest at $(0,0)$. $\quad T(0,0)=100$.

(c) $\text{grad}\, T(x,y)=\dfrac{-200x}{(1+x^2+y^2)^2}\vec{i}-\dfrac{200y}{(1+x^2+y^2)^2}\vec{j}$

$\text{grad}\, T(3,2)=\dfrac{-600}{14^2}\vec{i}-\dfrac{400}{14^2}\vec{j}$ This vector points in the direction of maximum rate of change.

The maximum rate of change is $\|\text{grad}\, T(3,2)\|=\sqrt{\left(\frac{600}{14^2}\right)^2+\left(\frac{400}{14^2}\right)^2}=\dfrac{50\sqrt{13}}{49}$

(d) Direction: $-\text{grad}\, T(3,2)=\dfrac{600}{14^2}\vec{i}+\dfrac{600}{14^2}\vec{j}$

(e) Any vector perpendicular to the gradient; for example, $4\vec{i}-6\vec{j}$.

$\boxed{14.4/64}$ The correct answer is (b).

Let $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$. Then the rate of change of the temperature with respect to distance in the direction that the bug is moving is $H_{\vec{u}}(x,y) = \text{grad}\, H(x,y) \cdot \vec{u}$.

But the rate of change of distance with respect to time for the bug is $\|\vec{v}\|$ (ie the speed of the bug).

The rate of change of the temperature with respect to time is then

$$\left(\text{grad}\, H(x,y) \cdot \vec{u}\right) \|\vec{v}\| = \left(\text{grad}\, H(x,y) \cdot \frac{\vec{v}}{\cancel{\|\vec{v}\|}}\right) \cancel{\|\vec{v}\|}$$

$$= \boxed{\text{grad}\, H(x,y) \cdot \vec{v}}$$

$\boxed{14.5/8}$

$$f(p,q,r) = e^p + \ln q + e^{r^2}$$

$$\text{grad } f(p,q,r) = f_p(p,q,r)\vec{i} + f_q(p,q,r)\vec{j} + f_r(p,q,r)\vec{k}$$

$$= (e^p)\vec{i} + \left(\frac{1}{q}\right)\vec{j} + \left(2re^{r^2}\right)\vec{k}$$

$\boxed{14.5/12}$

$$f(x,y,z) = xy + z^2$$

$$\text{grad } f(x,y,z) = y\vec{i} + x\vec{j} + 2z\vec{k}$$

$$\text{grad } f(1,1,1) = \vec{i} + \vec{j} + 2\vec{k}$$

The direction given is $\vec{v} = \vec{i} + 2\vec{j} + 3\vec{k}$, and a <u>unit vector</u> in this direction is $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{14}}(\vec{i} + 2\vec{j} + 3\vec{k})$

Then $f_{\vec{u}}(1,1,1) = \text{grad } f(1,1,1) \bullet \vec{u}$

$$= (\vec{i} + \vec{j} + 2\vec{k}) \bullet \left[\frac{1}{\sqrt{14}}(\vec{i} + 2\vec{j} + 3\vec{k})\right]$$

$$= \boxed{\frac{9}{\sqrt{14}}}$$

$\boxed{14.5/21}$ Check: $\cos(-1+1) = \cos(0) = 1$, and $e^{-2+2} = e^0 = 1$, so $(-1,1,2)$ is on the surface.

Rewrite the equation as $\cos(x+y) - e^{xz+2} = 0$, so the surface is a level surface of $f(x,y,z) = \cos(x+y) - e^{xz+2}$. Then

$$\text{grad } f(x,y,z) = (\sin(x+y) - ze^{xz+2})\vec{i} + (\sin(x+y))\vec{j} + (-xe^{xz+2})\vec{k},$$

and $\vec{n} = \text{grad } f(-1,1,2) = -2\vec{i} + \vec{k}$.

The tangent plane is

$$-2(x+1) + (z-2) = 0$$

or $$z = 2x + 4$$

$\boxed{14.5/26}$ Let $f(x,y,z) = x^2 + y^2 + 3z^2$, so the surface is the level surface $f(x,y,z) = 4$.

(a) $\text{grad}\, f(x,y,z) = 2x\vec{i} + 2y\vec{j} + 6z\vec{k}$

$\vec{n} = \text{grad}\, f(0.6, 0.8, 1) = 1.2\vec{i} + 1.6\vec{j} + 6\vec{k}$

The tangent plane is

$$1.2(x-0.6) + 1.6(y-0.8) + 6(z-1) = 0$$

(b) A vector normal to the given plane is $8\vec{i} + 6\vec{j} + 30\vec{k}$.

We want a point (x,y,z) on the surface where the normal vector is <u>parallel to</u> $8\vec{i} + 6\vec{j} + 30\vec{k}$.

That is, we want $\text{grad}\, f(x,y,z) = \lambda(8\vec{i} + 6\vec{j} + 30\vec{k})$,

where λ is a constant to be determined.

Thus we must find points <u>on the surface</u> where

$$2x\vec{i} + 2y\vec{j} + 6z\vec{k} = 8\lambda\vec{i} + 6\lambda\vec{j} + 30\lambda\vec{k}$$

$$\Rightarrow \begin{cases} 2x = 8\lambda \\ 2y = 6\lambda \\ 6z = 30\lambda \end{cases} \Rightarrow \begin{cases} x = 4\lambda \\ y = 3\lambda \\ z = 5\lambda \end{cases}$$

Put this into $x^2 + y^2 + 3z^2 = 4$

to get $16\lambda^2 + 9\lambda^2 + 75\lambda^2 = 4$

$$\lambda^2 = \frac{4}{100} = \frac{1}{25}$$

$$\underline{\lambda = \pm\frac{1}{5}}$$

If $\lambda = \frac{1}{5}$, we have $x = \frac{4}{5}$, $y = \frac{3}{5}$, $z = 1$.

If $\lambda = -\frac{1}{5}$, we have $x = -\frac{4}{5}$, $y = -\frac{3}{5}$, $z = -1$.

So there are <u>two</u> points where the tangent plane is parallel to the given plane.

$\boxed{14.5/30}$

(a) grad $f(1,3)$ is normal to the level curve through $(1,3)$, so $\vec{n} = 2\vec{i} - 5\vec{j}$, and the tangent line is

$$2(x-1) - 5(y-3) = 0$$

or $\quad 2x - 5y = -13$

(b) Since grad $f(1,3) = 2\vec{i} - 5\vec{j}$, we know $f_x(1,3) = 2$ and $f_y(1,3) = -5$.

The tangent plane is

$$z = f(1,3) + f_x(1,3)(x-1) + f_y(1,3)(y-3)$$

$$z = 7 + 2(x-1) - 5(y-3)$$

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$\boxed{14.5/36}$

$$\varphi(x,y,z) = \frac{GMm}{\|\vec{r}\|} = \frac{GMm}{\sqrt{x^2+y^2+z^2}}$$

$$\text{grad}\,\varphi(x,y,z) = \frac{-GMm\,x}{(x^2+y^2+z^2)^{3/2}}\vec{i} + \frac{-GMm\,y}{(x^2+y^2+z^2)^{3/2}}\vec{j} + \frac{-GMm\,y}{(x^2+y^2+z^2)^{3/2}}\vec{k}$$

$$= \frac{-GMm}{\left[(x^2+y^2+z^2)^{1/2}\right]^3}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \frac{-GMm}{\|\vec{r}\|^3}\vec{r} = \vec{F}$$

$\boxed{14.5/32}$

$G(x,y,z) = x^2 - 5xy + y^2 z$

$\text{grad } G(x,y,z) = (2x-5y)\vec{i} + (-5x+2yz)\vec{j} + (y^2)\vec{k}$

(a) $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{1}{\sqrt{21}}(2\vec{i}+\vec{j}-4\vec{k})$

$G_{\vec{u}}(1,2,3) = \text{grad } G(1,2,3) \bullet \vec{u}$

$= (-8\vec{i} + 7\vec{j} + 4\vec{k}) \bullet \left(\dfrac{1}{\sqrt{21}}(2\vec{i}+\vec{j}-4\vec{k})\right)$

$= \dfrac{-16+7-16}{\sqrt{21}} = \dfrac{-25}{\sqrt{21}}$

(b) The gradient points in the direction of maximum rate of change: $\text{grad } G(1,2,3) = \underline{-8\vec{i} + 7\vec{j} + 4\vec{k}}$

(c) $\|\text{grad } G(1,2,3)\| = \sqrt{64+49+16} = \sqrt{129}$