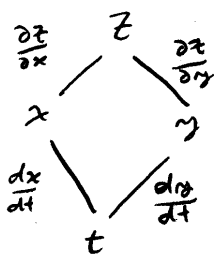
$\boxed{14.6/2}$ $z = \ln(x^2+y^2)$, $x = \frac{1}{t}$, $y = \sqrt{t} = t^{\frac{1}{2}}$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

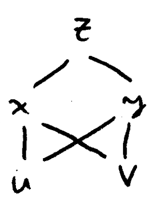
$$= \left(\frac{2x}{x^2+y^2}\right)\left(\frac{-1}{t^2}\right) + \left(\frac{2y}{x^2+y^2}\right)\left(\frac{1}{2}t^{-\frac{1}{2}}\right)$$

$\Downarrow$ (Put in $x = \frac{1}{t}$ and $y = t^{\frac{1}{2}}$)

$$= \left(\frac{\frac{2}{t}}{\frac{1}{t^2}+t}\right)\left(\frac{-1}{t^2}\right) + \left(\frac{\cancel{2}t^{\frac{1}{2}}}{\frac{1}{t^2}+t}\right)\left(\cancel{\tfrac{1}{2}}\cancel{t^{-\frac{1}{2}}}\right)$$

$$= \frac{-2}{t+t^4} + \frac{t^3}{t+t^4} = \boxed{\frac{t^3-2}{t^4+t}}$$

$\boxed{14.6/10}$ $z = (x+y)e^y$, $x = u^2+v^2$, $y = u^2-v^2$



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= (e^y)(2u) + \left((x+y)e^y + e^y\right)(2u)$$

$$= 2ue^y(2 + x + y)$$

$$= 2ue^{u^2-v^2}(2 + u^2 + \cancel{v^2} + u^2 - \cancel{v^2}) = \boxed{4ue^{(u^2-v^2)}(1+u^2)}$$

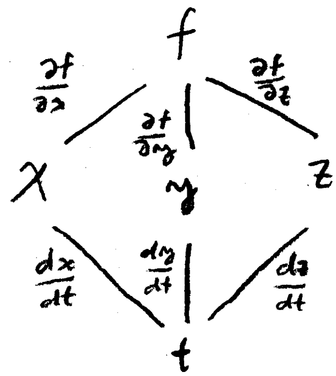
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (e^y)(2v) + \left((x+y)e^y + e^y\right)(-2v)$$

$$= \cancel{2ve^y} + (-2v)(x+y)e^y - \cancel{2ve^y}$$

$$= -2v(u^2 \cancel{+v^2} + u^2 \cancel{-v^2})e^{u^2-v^2}$$

$$= \boxed{-4u^2ve^{(u^2-v^2)}}$$

$\boxed{14.6/16}$



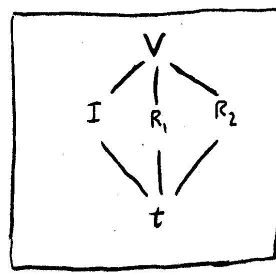
$$\boxed{\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}}$$

$$\left(\text{or } \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}\right)$$

$\boxed{14.6/19}$ We have $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$, so $R = \frac{R_1 R_2}{R_1 + R_2}$

Then $V = IR = I\frac{R_1 R_2}{R_1 + R_2}$, and by the chain rule,

$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R_1}\frac{dR_1}{dt} + \frac{\partial V}{\partial R_2}\frac{dR_2}{dt}$$



Now $\frac{\partial V}{\partial I} = \frac{R_1 R_2}{R_1 + R_2}$, $\frac{\partial V}{\partial R_1} = I\left(\frac{(R_1+R_2)R_2 - R_1 R_2}{(R_1+R_2)^2}\right) = \frac{IR_2^2}{(R_1+R_2)^2}$

$$\frac{\partial V}{\partial R_2} = I\left(\frac{(R_1+R_2)R_1 - R_1 R_2}{(R_1+R_2)^2}\right) = \frac{IR_1^2}{(R_1+R_2)^2}$$

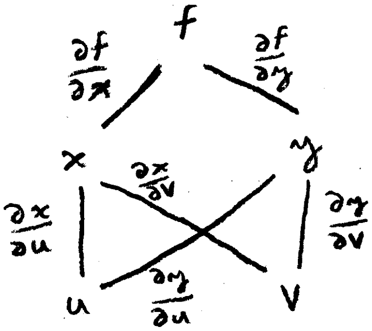
Using the data given, we have

$$\frac{\partial V}{\partial I} = \frac{(3)(5)}{3+5} = \frac{15}{8} \quad \frac{\partial V}{\partial R_1} = \frac{(2)(5^2)}{(3+5)^2} = \frac{50}{64}, \quad \frac{\partial V}{\partial R_2} = \frac{(2)(3)^2}{(3+5)^2} = \frac{18}{64}$$

and $\frac{dI}{dt} = 10^{-2}$, $\frac{dR_1}{dt} = 0.5$, $\frac{dR_2}{dt} = -0.1$

so $\frac{dV}{dt} = \left(\frac{15}{8}\right)\left(10^{-2}\right) + \left(\frac{50}{64}\right)(0.5) + \left(\frac{18}{64}\right)(-0.1) = 0.38125$ volts/sec

$\boxed{14.6/21}$



If $z = f(x,y)$, and $x = x(u,v)$, $y = y(u,v)$,

then
$$z_u = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$
$$= f_x(x,y)\,x_u(u,v) + f_y(x,y)\,y_u(u,v)$$

(This means we're interpreting z as
$\underline{z(u,v) = f(x(u,v), y(u,v))}$)

So
$$z_u(1,2) = f_x\big(x(1,2), y(1,2)\big)\,x_u(1,2) + f_y\big(x(1,2), y(1,2)\big)\,y_u(1,2)$$

$(u=1,\ v=2)$

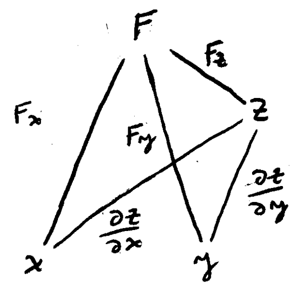
$$= f_x(5,3)\,x_u(1,2) + f_y(5,3)\,y_u(1,2)$$
$$= be + dp$$

$\boxed{14.6/26}$

Let $w(x,y) = F(x,y,f(x,y))$

Since $F(x,y,f(x,y)) = 0$, we know $\frac{\partial w}{\partial x} = 0$ and $\frac{\partial w}{\partial y} = 0$.

we have



so by the chain rule,

$$\frac{\partial w}{\partial x} = F_x(x,y,f(x,y)) + F_z(x,y,f(x,y))\frac{\partial z}{\partial x}$$

since $\frac{\partial w}{\partial x} = 0$, we have

$$F_x(x,y,f(x,y)) + F_z(x,y,f(x,y))\frac{\partial z}{\partial x} = 0,$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-F_x(x,y,f(x,y))}{F_z(x,y,f(x,y))}, \text{ i.e. } \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

Similarly,

$$0 = \frac{\partial w}{\partial y} = F_y(x,y,f(x,y)) + F_z(x,y,f(x,y))\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{-F_y(x,y,f(x,y))}{F_z(x,y,f(x,y))}, \quad \text{i.e. } \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

14.6/27 **(a)** We will use the chain rule identities,

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \quad\text{and}\quad \frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}.$$

These equations are to be in terms of $\partial z/\partial x$ and $\partial z/\partial y$, so we may calculate the other terms, switching from Cartesian to polar coordinates. Recall polar coordinates :

$$x=r\cos\theta, \quad y=r\sin\theta$$

Thus we have

$$\begin{aligned}
\frac{\partial x}{\partial r}&=\frac{\partial(r\cos\theta)}{\partial r}=\cos\theta\\
\frac{\partial y}{\partial r}&=\frac{\partial(r\sin\theta)}{\partial r}=\sin\theta\\
\frac{\partial x}{\partial \theta}&=\frac{\partial(r\cos\theta)}{\partial \theta}=-r\sin\theta\\
\frac{\partial y}{\partial \theta}&=\frac{\partial(r\sin\theta)}{\partial \theta}=r\cos\theta
\end{aligned}$$

Now, substituting into the equations for $\partial z/\partial r$ and $\partial z/\partial \theta$, we get

$$(1)\qquad \frac{\partial z}{\partial r}=\cos\theta\frac{\partial z}{\partial x}+\sin\theta\frac{\partial z}{\partial y}$$

$$(2)\qquad \frac{\partial z}{\partial \theta}=-r\sin\theta\frac{\partial z}{\partial x}+r\cos\theta\frac{\partial z}{\partial y}.$$

We will call these equations (1) and (2).

(b) Now we solve for $\partial z/\partial x$ and $\partial z/\partial y$. From (2) we get:

$$(3)\qquad \frac{\partial z}{\partial x}=\left(\frac{\partial z}{\partial \theta}-r\cos\theta\frac{\partial z}{\partial y}\right)\left(\frac{-1}{r\sin\theta}\right),$$

Now substitute (3) into (1):

$$\begin{aligned}
\frac{\partial z}{\partial r}&=\cos\theta\left(\frac{\partial z}{\partial \theta}-r\cos\theta\frac{\partial z}{\partial y}\right)\left(\frac{-1}{r\sin\theta}\right)+\sin\theta\frac{\partial z}{\partial y}\\
&=-\frac{\cos\theta}{r\sin\theta}\frac{\partial z}{\partial \theta}+\frac{\cos^2\theta}{\sin\theta}\frac{\partial z}{\partial y}+\sin\theta\frac{\partial z}{\partial y}
\end{aligned}$$

Now solve for $\partial z/\partial y$:

$$\begin{aligned}
\frac{\partial z}{\partial y}\left(\frac{\cos^2\theta}{\sin\theta}+\frac{\sin^2\theta}{\sin\theta}\right)&=\frac{\partial z}{\partial r}+\frac{\cos\theta}{r\sin\theta}\frac{\partial z}{\partial \theta}\\
\frac{\partial z}{\partial y}\left(\frac{1}{\sin\theta}\right)&=\frac{\partial z}{\partial r}+\frac{\cos\theta}{r\sin\theta}\frac{\partial z}{\partial \theta}\\
\frac{\partial z}{\partial y}&=\sin\theta\frac{\partial z}{\partial r}+\frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}.
\end{aligned}$$

Now, substitute $\partial z/\partial y$ into equation (3) and solve for $\partial z/\partial x$.

$$\begin{aligned}
\frac{\partial z}{\partial x}&=\left(\frac{\partial z}{\partial \theta}-r\cos\theta\frac{\partial z}{\partial y}\right)\frac{-1}{r\sin\theta}\\
&=\frac{-1}{r\sin\theta}\frac{\partial z}{\partial \theta}+\frac{\cos\theta}{\sin\theta}\left(\sin\theta\frac{\partial z}{\partial r}+\frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}\right)\\
&=\cos\theta\frac{\partial z}{\partial r}+\frac{\cos^2\theta-1}{r\sin\theta}\frac{\partial z}{\partial \theta}\\
&=\cos\theta\frac{\partial z}{\partial r}-\frac{\sin^2\theta}{r\sin\theta}\frac{\partial z}{\partial \theta}\\
&=\cos\theta\frac{\partial z}{\partial r}-\frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}.
\end{aligned}$$

(c) Now we use the chain rule to get $\partial z/\partial x$ and $\partial z/\partial y$.

$$(4) \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y}, \qquad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x}$$

We will call this equation (4).
As before, we will calculate some of these partials using $r = \sqrt{x^2+y^2}$ and $\theta = \arctan(y/x)$

$$\frac{\partial r}{\partial y} = \frac{\partial\sqrt{x^2+y^2}}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta$$
$$\frac{\partial \theta}{\partial y} = \frac{\partial \arctan(y/x)}{\partial y} = \frac{1}{1+(\frac{y}{x})^2}(x^{-1}) = \frac{x}{(x^2+y^2)} = \frac{\cos\theta}{r}$$
$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta$$
$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(\frac{y}{x})^2}(-\frac{y}{x^2}) = -\frac{y}{(x^2+y^2)} = -\frac{\sin\theta}{r}$$

Now, substituting these into (4), we get:

$$\frac{\partial z}{\partial y} = \sin\theta\frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}$$
$$\frac{\partial z}{\partial x} = \cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}$$

Note that these equations match with those found in part (b).

28 ~~24.~~ Using $x = r\cos\theta$ and $y = r\sin\theta$ we compute $\partial z/\partial r$ and $\partial z/\partial\theta$ in terms of $\partial z/\partial x$ and $\partial z/\partial y$:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$
$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}r\cos\theta$$

So we have

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta$$

In addition we have,

$$\frac{1}{r}\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}(-\sin\theta) + \frac{\partial z}{\partial y}\cos\theta$$

thus,

$$\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta$$

Adding we get

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

~~25.~~ Since $\left(\frac{\partial U}{\partial P}\right)_V$ involves the variables P and V, we are viewing U as a function of these two variables, so $U = U_3(P, V)$.
Then

$$\left(\frac{\partial U}{\partial P}\right)_V = \frac{\partial U_3(P, V)}{\partial P}.$$

~~26.~~ To calculate $\left(\frac{\partial U}{\partial P}\right)_T$, we think of U as a function of P and T, as in $U_1(T, P)$. Thus

$$\left(\frac{\partial U}{\partial P}\right)_T = \frac{\partial U_1}{\partial P}.$$

$\boxed{14.7/16}$ $f(x,y) = \cos(x+3y)$

$f(0,0) = \cos(0) = 1$

$f_x(x,y) = -\sin(x+3y)$ $\quad$ $f_x(0,0) = 0$

$f_y(x,y) = -3\sin(x+3y)$ $\quad$ $f_y(0,0) = 0$

$f_{xx}(x,y) = -\cos(x+3y)$ $\quad$ $f_{xx}(0,0) = -1$

$f_{xy}(x,y) = -3\cos(x+3y)$ $\quad$ $f_{xy}(0,0) = -3$

$f_{yy}(x,y) = -9\cos(x+3y)$ $\quad$ $f_{yy}(0,0) = -9$

$Q(x,y) = 1 - \frac{1}{2}x^2 - 3xy - \frac{9}{2}y^2$

NOT ASSIGNED

$\boxed{14.7/19}$

(a) $f_x(P) > 0$ (b) $f_y(P) = 0$ (c) $f_{xx}(P) > 0$

(d) $f_{yy}(P) = 0$ (e) $f_{xy}(P) = 0$

$\boxed{14.7/20}$

(a) $f_x(P) > 0$ (b) $f_y(P) = 0$ (c) $f_{xx}(P) < 0$

(d) $f_{yy}(P) = 0$ (e) $f_{xy}(P) = 0$

NOT ASSIGNED

$\boxed{14.7/25}$

(a) $f_x(P) < 0$ (b) $f_y(P) < 0$ (c) $f_{xx}(P) = 0$

(d) $f_{yy}(P) = 0$ (e) $f_{xy}(P) = 0$

$\boxed{14.7/27}$

(a) $f_x(P) > 0$ (b) $f_y(P) < 0$ (c) $f_{xx}(P) < 0$

(d) $f_{yy}(P) < 0$ (e) $f_{xy}(P) > 0$

$\boxed{14.7/28}$

(a) $f_x(P) < 0$ (b) $f_y(P) > 0$ (c) $f_{xx}(P) > 0$

(d) $f_{yy}(P) > 0$ (e) $f_{xy}(P) < 0$

$\boxed{14.7/31}$

$f(x,y) = xe^{-y}$ $\qquad f(1,0) = 1$

$f_x(x,y) = e^{-y}$ $\qquad f_x(1,0) = 1$

$f_y(x,y) = -xe^{-y}$ $\qquad f_y(1,0) = -1$

$f_{xx}(x,y) = 0$ $\qquad f_{xx}(1,0) = 0$

$f_{yy}(x,y) = xe^{-y}$ $\qquad f_{yy}(1,0) = 1$

$f_{xy}(x,y) = -e^{-y}$ $\qquad f_{xy}(1,0) = -1$

$L(x,y) = 1 + (1)(x-1) + (-1)(y) = x - y$

$Q(x,y) = L(x,y) + \frac{f_{xx}(1,0)}{2}(x-1)^2 + f_{xy}(1,0)(x-1)(y) + \frac{f_{yy}(1,0)}{2}(y)^2$

$= x - y - (x-1)y + \frac{1}{2}y^2$

$L(0.9, 0.2) = 0.7$

$Q(0.9, 0.2) = 0.74$

$f(0.9, 0.2) = 0.7368\ldots$

We see that the quadratic approximation is significantly better than the linear approximation.

14.7/35

(a) Moving parallel to the x-axis means that the z-labels on the contours increase, so z is an increasing function of x. Moving parallel to the y-axis, the z-labels decrease, so z is a decreasing function of y.
(b) Since z is an increasing function of x, we have $f_x > 0$. Similarly, $f_y < 0$.
(c) Since the contours get closer together as we move parallel to the x-axis, we have $f_{xx} > 0$. This means that z is increasing faster and faster as x increases. Similar reasoning shows that $f_{yy} < 0$.
(d) The vector grad f is perpendicular to the level curves and points in the direction of increasing f values. See Figure 14.19.
(e) The vector grad f is longer at P because the contours are closer together at P than at Q.

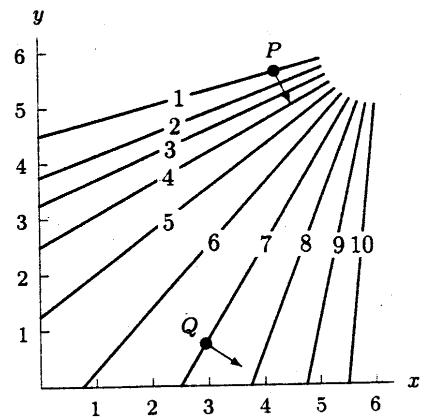


Figure 14.19

14.7/36

(a) Since P and Q lie on the same level curve, we have $a = k$.
(b) We have $b = f_x$ and $c = f_y$. Since the gradient of f at P (respectively Q) points towards M or away from M, from the figure, we see $f_x(P)$ and $f_y(P)$ have opposite signs, while $f_x(Q)$ and $f_y(Q)$ have the same signs. Thus Q is the point (x_1, y_1), so P is (x_2, y_2).
(c) Since $b = f_x(Q) > 0$ and $c = f_y(Q) > 0$, the value of f must increase as we go away from M. Thus, M must be a minimum (the surface is a valley).
(d) Since M is a minimum, $m = f_x(P) < 0$ and $n = f_y(P) > 0$.

~~32~~ (a) Calculate the partial derivatives:

$$\begin{array}{lll}
f(x,y) = \sin x \sin y & f(0,0) = 0 & f(\frac{\pi}{2}, \frac{\pi}{2}) = 1 \\
f_x(x,y) = \cos x \sin y & f_x(0,0) = 0 & f_x(\frac{\pi}{2}, \frac{\pi}{2}) = 0 \\
f_y(x,y) = \sin x \cos y & f_y(0,0) = 0 & f_y(\frac{\pi}{2}, \frac{\pi}{2}) = 0 \\
f_{xx}(x,y) = -\sin x \sin y & f_{xx}(0,0) = 0 & f_{xx}(\frac{\pi}{2}, \frac{\pi}{2}) = -1 \\
f_{xy}(x,y) = \cos x \cos y & f_{xy}(0,0) = 1 & f_{xy}(\frac{\pi}{2}, \frac{\pi}{2}) = 0 \\
f_{yy}(x,y) = -\sin x \sin y & f_{yy}(0,0) = 0 & f_{yy}(\frac{\pi}{2}, \frac{\pi}{2}) = -1
\end{array}$$

Thus, the Taylor polynomial about $(0,0)$ is

$$f(x,y) \approx Q_1(x,y) = xy.$$

The Taylor polynomial about $(\frac{\pi}{2}, \frac{\pi}{2})$ is

$$f(x,y) \approx Q_2(x,y) = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2.$$