$\boxed{15.1/8}$

$f(x,y) = (x+y)(xy+1)$

$f_x(x,y) = (x+y)y + (xy+1) = 2xy + y^2 + 1$

$f_y(x,y) = (x+y)x + (xy+1) = 2xy + x^2 + 1$

f is differentiable for all (x,y), so the only critical points are where $f_x = 0$ and $f_y = 0$.

$\Rightarrow$ ① $2xy + y^2 + 1 = 0$ and ② $2xy + x^2 + 1 = 0$

subtract: $y^2 - x^2 = 0 \Rightarrow y = x$ OR $y = -x$

If $y = x$, then ① implies $3y^2 = -1$, which has no solutions.

If $y = -x$, then ① implies $-2y^2 + y^2 + 1 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

If $y = 1$, then $x = -1$. If $y = -1$, then $x = 1$.

So the critical points are $(-1,1)$ and $(1,-1)$.

To apply the second derivative test, we need

$f_{xx}(x,y) = 2y$, $f_{xy}(x,y) = 2x + 2y$, $f_{yy}(x,y) = 2x$.

At $(-1,1)$,

$D = f_{xx}(-1,1) f_{yy}(-1,1) - \left(f_{xy}(-1,1)\right)^2 = (2)(-2) - 0^2 = -4 < 0$,

so $\underline{(-1,1) \text{ is a saddle point.}}$

At $(1,-1)$,

$D = f_{xx}(1,-1) f_{yy}(1,-1) - \left(f_{xy}(1,-1)\right)^2 = (-2)(2) - 0^2 = -4 < 0$,

so $\underline{(1,-1) \text{ is also a saddle point.}}$

16. Figure 15.2 shows the direction of ∇f at the points where $||\nabla f||$ is largest, since at those points the contours are closest together.

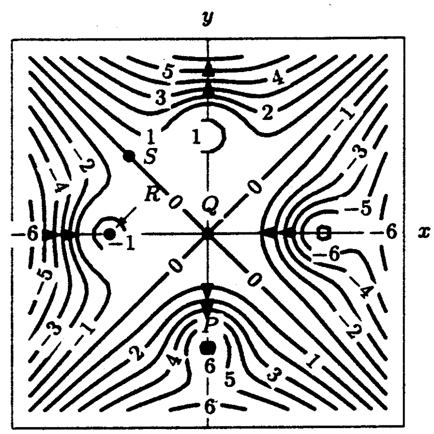


Figure 15.2

17. To find critical points, set partial derivatives equal to zero:

$$E_x = \sin x = 0 \quad \text{when} \quad x = 0,\ \pm\pi,\ \pm 2\pi, \cdots$$

$$E_y = y = 0 \quad \text{when} \quad y = 0.$$

The critical points are

$$\cdots(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0), (3\pi, 0)\cdots$$

To classify, calculate $D = E_{xx}E_{yy} - (E_{xy})^2 = \cos x$.
At the points $(0,0), (\pm 2\pi, 0), (\pm 4\pi, 0), (\pm 6\pi, 0), \cdots$

$$D = (1) > 0 \quad \text{and} \quad E_{xx} > 0 \quad (\text{Since} E_{xx}(0, 2k\pi) = \cos(2k\pi) = 1).$$

Therefore $(0,0), (\pm 2\pi, 0), (\pm 4\pi, 0), (\pm 6\pi, 0), \cdots$ are local minima.
At the points $(\pm\pi, 0), (\pm 3\pi, 0), (\pm 5\pi, 0), (\pm 7\pi, 0), \cdots$, we have $\cos(2k+1)\pi = -1$, so

$$D = (-1) < 0.$$

Therefore $(\pm\pi, 0), (\pm 3\pi, 0), (\pm 5\pi, 0), (\pm 7\pi, 0), \cdots$ are saddle points.

15.1/22 [redacted] At a critical point,

$$f_x = \cos x \sin y = 0 \quad \text{so} \quad \cos x = 0 \text{ or } \sin y = 0;$$

and

$$f_y = \sin x \cos y = 0 \quad \text{so} \quad \sin x = 0 \text{ or } \cos y = 0.$$

Case 1: Assume $\cos x = 0$. This gives

$$x = \cdots - \frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$$

(This can be written more compactly as: $x = k\pi + \pi/2$, for $k = 0, \pm 1, \pm 2, \cdots$.)
If $\cos x = 0$, then $\sin x = \pm 1 \neq 0$. Thus in order to have $f_y = 0$ we need $\cos y = 0$, giving

$$y = \cdots - \frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$$

(More compactly, $y = l\pi + \pi/2$, for $l = 0, \pm 1, \pm 2, \cdots$)
Case 2: Assume $\sin y = 0$. This gives

$$y = \cdots - 2\pi, -\pi, 0, \pi, 2\pi, \cdots$$

(More compactly, $y = l\pi$, for $l = 0, \pm 1, \pm 2, \cdots$)
If $\sin y = 0$, then $\cos y = \pm 1 \neq 0$, so to get $f_y = 0$ we need $\sin x = 0$, giving

$$x = \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$$

(More compactly, $x = k\pi$ for $k = 0, \pm 0, \pm 1, \pm 2, \cdots$)
Hence we get all the critical points of $f(x, y)$. Those from Case 1 are as follows:

$$\cdots\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)\cdots$$
$$\cdots\left(\frac{\pi}{2}, -\frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\cdots$$
$$\cdots\left(\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)\cdots$$

Those from Case 2 are as follows:

$$\cdots(-\pi, -\pi), (-\pi, 0), (-\pi, \pi), (-\pi, 2\pi)\cdots$$
$$\cdots(0, -\pi), (0, 0), (0, \pi), (0, 2\pi)\cdots$$
$$\cdots(\pi, -\pi), (\pi, 0), (\pi, \pi), (\pi, 2\pi)\cdots$$

More compactly these points can be written as,

$$(k\pi, l\pi), \text{ for } k = 0, \pm 1, \pm 2, \cdots, l = 0, \pm 1, \pm 2, \cdots$$
$$\text{and } (k\pi + \frac{\pi}{2}, l\pi + \frac{\pi}{2}), \text{ for } k = 0, \pm 1, \pm 2, \cdots, l = 0, \pm 1, \pm 2, \cdots$$

To classify the critical points, we find the discriminant. We have

$$f_{xx} = -\sin x \sin y, \qquad f_{yy} = -\sin x \sin y, \qquad \text{and} \qquad f_{xy} = \cos x \cos y.$$

Thus the discriminant is

$$\begin{aligned} D(x, y) &= f_{xx} f_{yy} - f_{xy}^2 \\ &= (-\sin x \sin y)(-\sin x \sin y) - (\cos x \cos y)^2 \\ &= \sin^2 x \sin^2 y - \cos^2 x \cos^2 y \\ &= \sin^2 y - \cos^2 x. \text{ (Use: } \sin^2 x = 1 - \cos^2 x \text{ and factor.)} \end{aligned}$$

At points of the form $(k\pi, l\pi)$ where $k = 0, \pm 1, \pm 2, \cdots; l = 0, \pm 1, \pm 2, \cdots$, we have
$D(x, y) = -1 < 0$ so $(k\pi, l\pi)$ are saddle points.
At points of the form $(k\pi + \frac{\pi}{2}, l\pi + \frac{\pi}{2})$ where $k = 0, \pm 1, \pm 2, \cdots; l = 0, \pm 1, \pm 2, \cdots$
$D(k\pi + \frac{\pi}{2}, l\pi + \frac{\pi}{2}) = 1 > 0$, so we have two cases:
If k and l are both even or k and l are both odd, then
$f_{xx} = -\sin x \sin y = -1 < 0$, so $(k\pi + \frac{\pi}{2}, l\pi + \frac{\pi}{2})$ are local maximum points.
If k is even but l is odd or k is odd but l is even, then
$f_{xx} = 1 > 0$ so $(k\pi + \frac{\pi}{2}, l\pi + \frac{\pi}{2})$ are local minimum points.

19. To find the critical points, we must solve the equations

$$\frac{\partial f}{\partial x} = e^x(1 - \cos y) = 0$$
$$\frac{\partial f}{\partial y} = e^x(\sin y) = 0.$$

The first equation has solution

$$y = 0, \pm 2\pi, \pm 4\pi, \ldots.$$

The second equation has solution

$$y = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots.$$

Since x can be anything, the lines

$$y = 0, \pm 2\pi, \pm 4\pi, \ldots$$

$\boxed{15.1/23}$

If $(-2,1)$ is a local maximum, it must be a critical point, so $f_x(-2,1) = 0$ and $f_y(-2,1) = 0$.

Now $f_x(x,y) = -2x - B$, $f_y(x,y) = -2y - C$

so $f_x(-2,1) = 4 - B = 0$ $\qquad$ $f_y(-2,1) = -2 - C = 0$

$B = 4$ $\qquad$ $C = -2$

We want $f(-2,1) = 15$, so

$$f(-2,1) = A - (4 - 8 + 1 - 2) = 15$$

$$A = 10$$

Thus $f(x,y) = 10 - (x^2 + 4x + y^2 - 2y)$

Check that it really is a local max:

$f_{xx}(x,y) = -2$, $f_{xy}(x,y) = 0$, $f_{yy}(x,y) = -2$

$f_{xx}(-2,1) = -2$, $f_{xy}(-2,1) = 0$, $f_{yy}(-2,1) = -2$

Then $D = (-2)(-2) - 0^2 = 4 > 0$

and $f_{xx}(-2,1) < 0$, so $(-2,1)$ is a local max.

(15.1/29 is on the next page)

15.1/31 The partial derivatives are

$$f_x(x,y) = 3x^2 - 3y^2 \qquad \text{and} \qquad f_y(x,y) = -6xy.$$

Now $f_x(x,y)$ will vanish if $x = \pm y$ and $f_y(x,y)$ will vanish if either $x = 0$ or $y = 0$. Since the partial derivatives are defined everywhere, the only critical points are where $f_x(x,y)$ and $f_y(x,y)$ vanish simultaneously. $(0,0)$ is the only critical point.

To find the contour for $f(x,y) = 0$, we solve the equation $x^3 - 3xy^2 = 0$. This can be factored into

$$f(x,y) = x(x - \sqrt{3}\,y)(x + \sqrt{3}\,y) = 0$$

whose roots are $x = 0$, $x = \sqrt{3}\,y$ and $x = -\sqrt{3}\,y$. Each of these roots describes a line through the origin; the three of them divide the plane into six regions. Crossing any one of these lines will change the sign of only one of the three factors of $f(x,y)$, which will change the sign of $f(x,y)$.

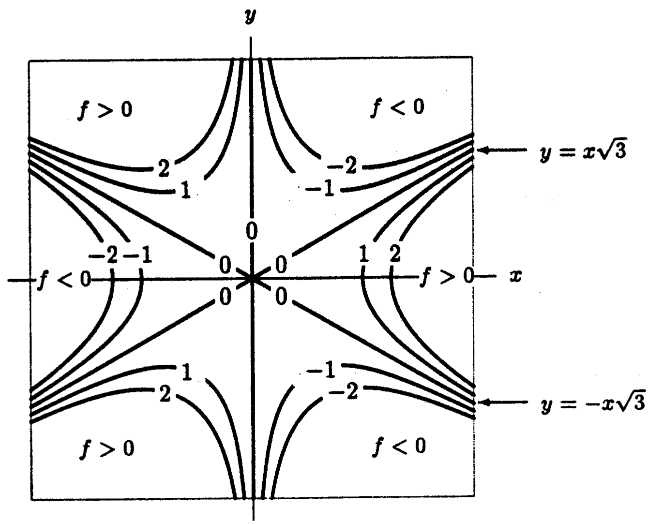


Figure 15.6

15.1/30 The first order partial derivatives are

$$f_x(x,y) = 2kx - 2y \qquad \text{and} \qquad f_y(x,y) = 2ky - 2x.$$

And the second order partial derivatives are

$$f_{xx}(x,y) = 2k \qquad f_{xy}(x,y) = -2 \qquad f_{yy}(x,y) = 2k$$

Since $f_x(0,0) = f_y(0,0) = 0$, the point $(0,0)$ is a critical point. The discriminant is

$$D = (2k)(2k) - 4 = 4(k^2 - 1).$$

For $k = \pm 2$, the discriminant is positive, $D = 12$. When $k = 2$, $f_{xx}(0,0) = 4$ which is positive so we have a local minimum at the origin. When $k = -2$, $f_{xx}(0,0) = -4$ so we have a local maximum at the origin. In the case $k = 0$, $D = -4$ so the origin is a saddle point.

Lastly, when $k = \pm 1$ the discriminant is zero, so the second derivative test can tell us nothing. Luckily, we can factor $f(x,y)$ when $k = \pm 1$. When $k = 1$,

$$f(x,y) = x^2 - 2xy + y^2 = (x - y)^2.$$

This is always greater than or equal to zero. So $f(0,0) = 0$ is a minimum and the surface is a trough-shaped parabolic cylinder with its base along the line $x = y$.

When $k = -1$,

$$f(x,y) = -x^2 - 2xy - y^2 = -(x + y)^2.$$

This is always less than or equal to zero. So $f(0,0) = 0$ is a maximum. The surface is a parabolic cylinder, with its top ridge along the line $x = -y$.

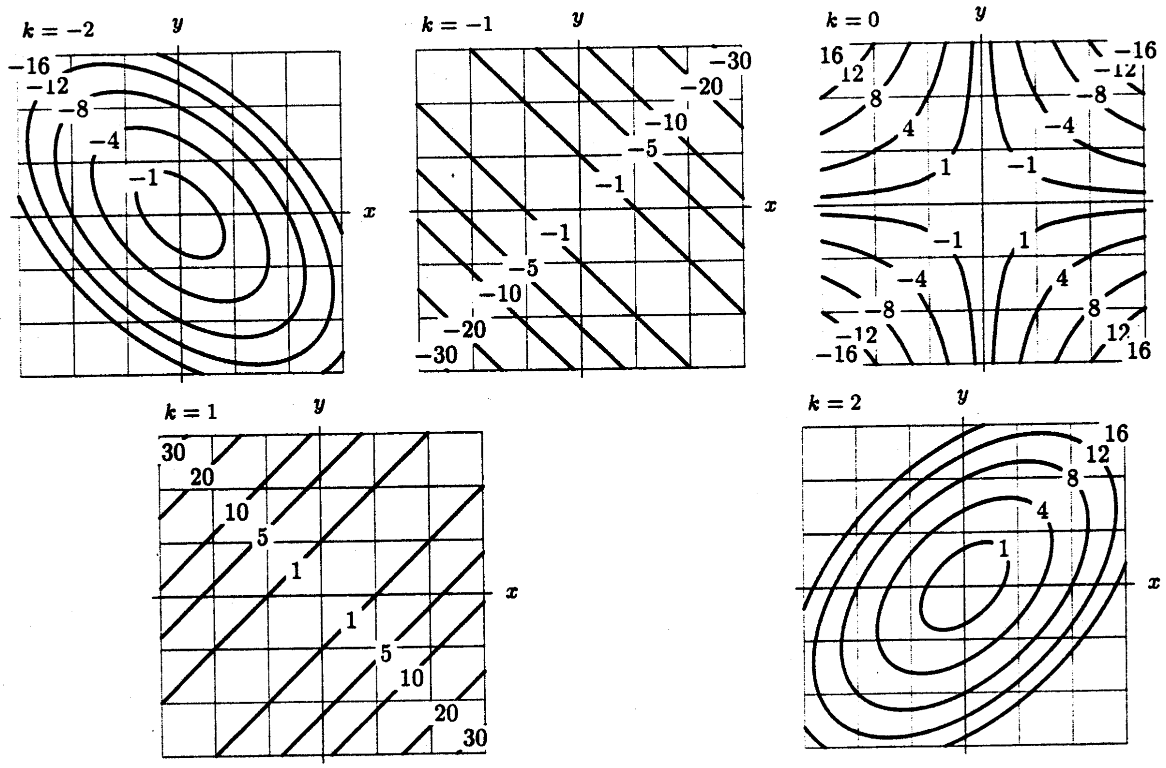


Figure 15.7

15.1/ ~~28~~. 29 We have

$$f_{xx}(x,y) = 2k, \quad f_{yy}(x,y) = 2, \quad f_{xy}(x,y) = -4,$$

so

$$D = f_{xx}(0,0)f_{yy}(0,0) - (f_{xy}(0,0))^2 = (2k)(2) - 16 = 4k - 16 = 4(k-4).$$

(a) If $k < 4$ then $D < 0$, and $(0,0)$ is a saddle point.

(b) Since $f_{yy} = 2 > 0$, $(0,0)$ is not a local maximum for any value of k.

(c) If $k > 4$ then $D > 0$, and since $f_{xx}(0,0) = 2k > 0$, $(0,0)$ is a local minimum.
What happens if $k = 4$? In this case, $f(x,y) = 4x^2 + y^2 - 4xy = (2x-y)^2$, so the graph of f is a parabolic cylinder that opens upwards. The minima occur along the line $2x - y = 0$. In this case, $(0,0)$ is also a local minimum (and so is any other point on the line $2x - y = 0$).

New York is split by a boundary between an 80s and a 70s region, so the northern portion of the state is likely to be about 74-76 while the southern portion is likely to be in the low 80s, maybe 81-84 or so.

California contains many different zones. The northern coastal areas will probably have the daily high as low as 65-68, although without another contour on that side, it is difficult to judge how quickly the temperature is dropping off to the west. The tip of Southern California is in a 100s region, so there we expect the daily high to be 100-101.

Arizona will have a low daily high around 85-87 in the northwest corner and a high in the 100s, perhaps 102-107 in its southern regions.

Massachusetts will probably have a high daily high around 81-84 and a low daily high of 70.

15.2/3,4,5

[redacted] The function f has no global maximum or global minimum; g has a global minimum (it is 0) but no global maximum; h has no global maximum or minimum.

[redacted] Since $f(x, y) \leq 0$ for all x, y and since $f(0, 0) = 0$, the function has a global maximum (it is 0) and no global minimum.

4. Suppose x is fixed. Then for large values of y the sign of f is determined by the highest power of y, namely y^3. Thus,

$$f(x, y) \to \infty \quad \text{as} \quad y \to \infty$$
$$f(x, y) \to -\infty \quad \text{as} \quad y \to -\infty.$$

So f does not have a global maximum or minimum.

5. To maximize $z = x^2 + y^2$, it suffices to maximize x^2 and y^2. We can maximize both of these at the same time by taking the point $(1, 1)$, where $z = 2$. It occurs on the boundary of the square. (Note: We also have maxima at the points $(-1, -1)$, $(-1, 1)$ and $(1, -1)$ which are on the boundary of the square.)

To minimize $z = x^2 + y^2$, we choose the point $(0, 0)$, where $z = 0$. It does not occur on the boundary of the square.

15.2/10

[redacted] To maximize this function, it suffices to maximize x^2 and minimize y^2. We can do this by choosing the point $(1, 0)$, or $(-1, 0)$ where $z = 1$. These occur on the boundary of the square.

To minimize $z = x^2 - y^2$, it suffices to maximize y^2 and minimize x^2. We can do this by taking the point $(0, 1)$, or $(0, -1)$ where $z = -1$. These occur on the boundary of the square.

Also see the next page.

7. To maximize $z = -x^2 - y^2$ it suffices to minimize x^2 and y^2. Thus, the maximum is at $(0, 0)$, where $z = 0$. It doesn't occur on the boundary of the square.

To minimize $z = -x^2 - y^2$, it suffices to maximize x^2 and y^2. Do this by taking the point $(1, 1)$, $(-1, -1)$, $(-1, 1)$, or $(1, -1)$ where $z = -2$. These occur on the boundary of the square.

15.2/8

[redacted] Let the line be in the form $y = b + mx$. When x equals $-1, 0$ and 1, then y equals $b - m$, b, and $b + m$, respectively. The sum of the squares of the vertical distances, which is what we want to minimize, is

$$f(m, b) = (2 - (b - m))^2 + (-1 - b)^2 + (1 - (b + m))^2.$$

To find the critical points, we compute the partial derivatives with respect to m and b,

$$\begin{aligned}
f_m &= 2(2 - b + m) + 0 + 2(1 - b - m)(-1) \\
&= 4 - 2b + 2m - 2 + 2b + 2m \\
&= 2 + 4m, \\
f_b &= 2(2 - b + m)(-1) + 2(-1 - b)(-1) + 2(1 - b - m)(-1) \\
&= -4 + 2b - 2m + 2 + 2b - 2 + 2b + 2m \\
&= -4 + 6b.
\end{aligned}$$

Setting both partial derivatives equal to zero, we get a system of equations:

$y = \frac{2}{3} - \frac{1}{2}x$

$$\begin{aligned}
2 + 4m &= 0, \\
-4 + 6b &= 0.
\end{aligned}$$

The solution is $m = -1/2$ and $b = 2/3$. One can check that it is a minimum. Hence, the regression line is $y = \frac{2}{3} - \frac{1}{2}x$.

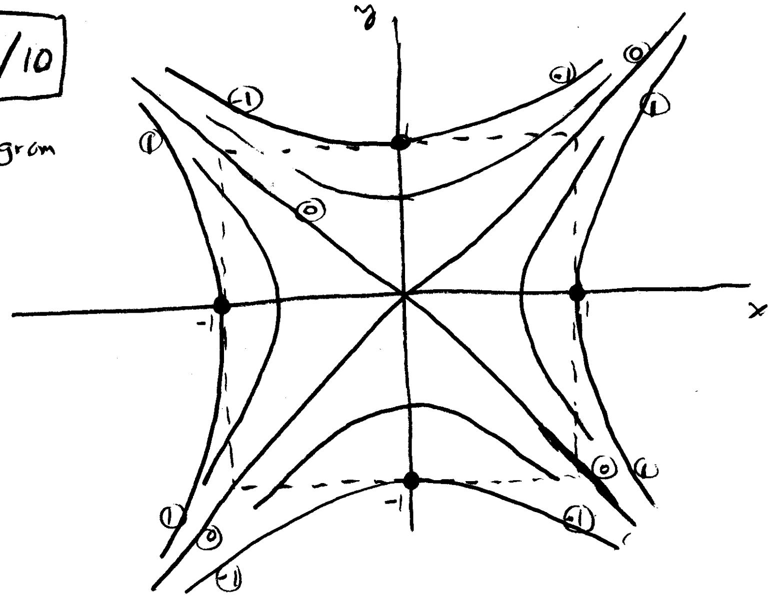
9. The maximum value, which is slightly above 30, say 30.5, occurs approximately at the origin. The minimum value, which is about 20.5, occurs at $(2.5, 5)$.

10. The maxima occur at about $(\pi/2, 0)$ and $(\pi/2, 2\pi)$. The minimum occurs at $(\pi/2, \pi)$. The maximum value is about 1, the minimum value is about -1.

11. The maximum value, which is about 11, occurs at $(5.1, 4.9)$. The minimum value, which is about -1, occurs at $(1, 3.9)$.

15.2/10

contour diagram for x^2-y^2 (a saddle)



The global max. is 1, and it occurs at $(1,0)$ and $(-1,0)$.
The global min. is -1, and it occurs at $(0,-1)$ and $(0,1)$.
(All the extreme points are on the boundary.)

Problems

The variables are a and b, so we set

$$\frac{\partial S}{\partial a} = 2(a+b) + 8(4a+b-2) + 18(9a+b-4) = 0$$
$$\frac{\partial S}{\partial b} = 2(a+b) + 2(4a+b-2) + 2(9a+b-4) = 0,$$

so, collecting terms and dividing by 4 and 2 respectively,

$$49a + 7b - 22 = 0$$
$$14a + 3b - 6 = 0.$$

Solving gives $a = 24/49$, $b = -2/7$.

Since there is only one critical point and S is unbounded as $a, b \to \infty$, this critical point is the global minimum. Therefore, the best fitting parabola is

$$y = \frac{24}{49}x^2 - \frac{2}{7}.$$

13. Let the sides be x, y, z cm. Then the volume is given by $V = xyz = 32$.

The surface area S is given by

$$S = 2xy + 2xz + 2yz.$$

Substituting $z = 32/(xy)$ gives

$$S = 2xy + \frac{64}{y} + \frac{64}{x}.$$

At a critical point,

$$\frac{\partial S}{\partial x} = 2y - \frac{64}{x^2} = 0$$
$$\frac{\partial S}{\partial y} = 2x - \frac{64}{y^2} = 0,$$

The symmetry of the equations (or by dividing the equations) tells us that $x = y$ and

$$2x - \frac{64}{x^2} = 0$$
$$x^3 = 32$$
$$x = 32^{1/3} = 3.17 \text{ cm}.$$

Thus the only critical point is $x = y = (32)^{1/3}$ cm and $z = 32/\left((32)^{1/3} \cdot (32)^{1/3}\right) = (32)^{1/3}$ cm. At the critical point

$$S_{xx}S_{yy} - (S_{xy})^2 = \frac{128}{x^3} \cdot \frac{128}{y^3} - 2^2 = \frac{(128)^2}{x^3y^3} - 4.$$

Since $D > 0$ and $S_{xx} > 0$ at this critical point, the critical point $x = y = z = (32)^{1/3}$ is a local minimum. Since $S \to \infty$ as $x, y \to \infty$, the local minimum is a global minimum.

15.2/24

Let the sides of the base be x and y cm. Let the height be z cm. Then the volume is given by $xyz = 32$ and the surface area, S, is given by

$$S = xy + 2xz + 2yz.$$

Substituting $z = 32/(xy)$ gives

$$S = xy + \frac{64}{y} + \frac{64}{x}.$$

At a critical point

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2} = 0$$
$$\frac{\partial S}{\partial y} = x - \frac{64}{y^2} = 0.$$

The symmetry of the equations tells us that $x = y$ and

$$\begin{aligned} x - \frac{64}{x^2} &= 0 \\ x^3 &= 64 \\ x &= 4 \text{ cm}. \end{aligned}$$

Thus the only critical point is $x = y = 4$ cm and $z = 32/(4 \cdot 4) = 2$ cm. At the critical point

$$D = S_{xx}S_{yy} - (S_{xy})^2 = \frac{128}{x^3} \cdot \frac{128}{y^3} - 1^2 = \frac{(128)^2}{x^3y^3} - 1.$$

Since $D > 0$ and $S_{xx} > 0$ at this critical point, the critical point $x = y = 4, z = 2$ is a local minimum. Since $S \to \infty$ as $x, y \to \infty$, the local minimum is a global minimum.

15. If the coordinates of the corner on the plane are (x, y, z), the volume of the box is $V = xyz$. Since $z = 1 - 3x - 2y$ on the plane, the volume is given by

$$V = xy(1 - 3x - 2y) = xy - 3x^2y - 2xy^2.$$

The domain is the triangular region $0 \le x \le \frac{1}{3}, 0 \le y \le (1 - 3x)/2$. At a critical point,

$$\begin{aligned} \frac{\partial V}{\partial x} &= y - 6xy - 2y^2 = y(1 - 6x - 2y) = 0 \\ \frac{\partial V}{\partial y} &= x - 3x^2 - 4xy = x(1 - 3x - 4y) = 0, \end{aligned}$$

One solution is $x = y = 0$. Another is $x = 0, y = \frac{1}{2}$; another is $y = 0, x = \frac{1}{3}$. Another is the solution of

$$\begin{aligned} 1 - 6x - 2y &= 0 \\ 1 - 3x - 4y &= 0, \end{aligned}$$

namely $x = \frac{1}{9}, y = \frac{1}{6}$.

If either $x = 0$ or $y = 0$, then $V = 0$, so these solutions do not give the maximum volume. Since

$$D = V_{xx}V_{yy} - (V_{xy})^2 = (-6y)(-4x) - (1 - 6x - 4y)^2$$

$$D\left(\frac{1}{9}, \frac{1}{6}\right) = \left(-6 \cdot \frac{1}{6}\right)\left(-4 \cdot \frac{1}{9}\right) - \left(1 - 6 \cdot \frac{1}{9} - 4 \cdot \frac{1}{6}\right)^2 = \frac{4}{9} - \frac{1}{9} = \frac{1}{3} > 0,$$

and $V_{xx}(\frac{1}{9}, \frac{1}{6}) = -1 < 0$, the point $x = \frac{1}{9}$, $y = \frac{1}{6}$, is a local maximum at which $V = (1/9)(1/6) - 3(1/9)^2(1/6) - 2(1/9)(1/6)^2 = 1/162$.

Since all points on the boundary of the domain give $V = 0$, the local maximum is a global maximum.

15.2/28 [redacted] The square of the distance from the point (x, y, z) to the origin is

$$S = x^2 + y^2 + z^2.$$

If the point is on the plane, $z = 1 - 3x - 2y$, we have

$$S = x^2 + y^2 + (1 - 3x - 2y)^2.$$

At the critical point

$$\begin{aligned} \frac{\partial S}{\partial x} &= 2x + 2(1 - 3x - 2y)(-3) = 2(10x + 6y - 3) = 0 \\ \frac{\partial S}{\partial y} &= 2y + 2(1 - 3x - 2y)(-2) = 2(6x + 5y - 2) = 0. \end{aligned}$$

Simplifying gives

$$\begin{aligned} 10x + 6y &= 3 \\ 6x + 5y &= 2, \end{aligned}$$

with solution $x = 3/14, y = 1/7$. At this point $z = 1/14$. We have

$$D = S_{xx}S_{yy} - (S_{xy})^2 = (20)(10) - 12^2 = 56.$$

so $D > 0$ and $S_{xx} > 0$. Thus, the point $x = 3/14, y = 1/7$ is a local minimum. Since $S \to \infty$ as $x, y \to \pm\infty$, the local minimum is a global minimum. Thus, $x = 3/14, y = 1/7, z = 1/14$ is the closest point to the origin on the plane.

Since $w, h \neq 0$, we can divide the first equation by the second giving

$$\frac{2wh^2}{2hw^2} = \frac{2048}{1024},$$

so

$$\frac{h}{w} = 2,$$

thus

$$h = 2w.$$

Substituting this in $C_h = 0$, we obtain $h^3 = 2048$, so $h = 12.7$ cm. Thus $w = h/2 = 6.35$ cm, and $l = 512/(wh) = 6.35$ cm. Now we check that these dimensions minimize the cost C. We find that

$$D = C_{hh}C_{ww} - C_{hw}^2 = (\frac{4096}{h^3})(\frac{2048}{w^3}) - 2^2,$$

and at $h = 12.7$, $w = 6.35$, $C_{hh} > 0$ and $D = 16 - 4 > 0$, thus C has a local minimum at $h = 12.7$ and $w = 6.35$. Since C increases without bound as $w, h \to 0$ or ∞, this local minimum must be a global minimum.

Therefore, the dimensions of the box that minimize the cost are $w = 6.35$ cm, $l = 6.35$ cm and $h = 12.7$ cm.

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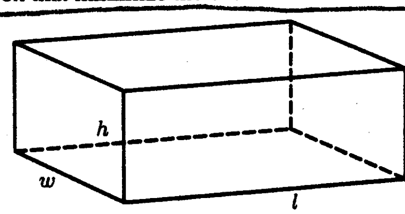

Figure 15.12

Let w, h and l be width, height and length of the suitcase in cm. Then its volume $V = lwh$, and $w + h + l \leq 135$. To maximize the volume V, choose $w + h + l = 135$, and thus $l = 135 - w - h$,

$$\begin{aligned} V &= wh(135 - w - h) \\ &= 135wh - w^2h - wh^2 \end{aligned}$$

Differentiating gives

$$\begin{aligned} V_w &= 135h - 2wh - h^2, \\ V_h &= 135w - w^2 - 2wh. \end{aligned}$$

Find the critical points by solving $V_w = 0$ and $V_h = 0$:

$$\begin{aligned} V_w = 0 \quad &\text{gives} \quad 135h - h^2 = 2wh, \\ V_h = 0 \quad &\text{gives} \quad 135w - w^2 = 2wh. \end{aligned}$$

As $hw \neq 0$, we cancel h (and w respectively) in the above equations and get

$$\begin{aligned} 135 - h &= 2w \\ 135 - w &= 2h \end{aligned}$$

Subtracting gives

$$w - h = 2(w - h)$$

hence $w = h$. Therefore, substituting into the equation $V_w = 0$

$$135h - h^2 = 2h^2$$

and therefore

$$3h^2 = 135h.$$

Since $h \neq 0$, we have
$$h = \frac{135}{3} = 45.$$
So $w = h = 45$ cm. Thus, $l = 135 - w - h = 45$ cm. To check that this critical point is a maximum, we find
$$V_{ww} = -2h, \quad V_{hh} = -2w,$$
$$V_{wh} = 135 - 2w - 2h,$$
so
$$D = V_{ww}V_{hh} - V_{wh}^2 = 4hw - (135 - 2w - 2h)^2.$$
At $w = h = 45$, we have $V_{ww} = -2(45) < 0$ and $D = 4(45)^2 - (135 - 90 - 90)^2 = 6075 > 0$, hence V is maximum at $w = h = l = 45$.

Therefore, the suitcase with maximum volume is a cube with dimensions width = height = length = 45 cm.

36. Let $P(K, L)$ be the profit obtained using K units of capital and L units of labor. The cost of production is given by
$$C(K, L) = kK + \ell L,$$
and the revenue function is given by
$$R(K, L) = pQ = pAK^aL^b.$$
Hence, the profit is
$$P = R - C = pAK^aL^b - (kK + \ell L).$$
In order to find local maxima of P, we calculate the partial derivatives and see where they are zero. We have:
$$\frac{\partial P}{\partial K} = apAK^{a-1}L^b - k,$$
$$\frac{\partial P}{\partial L} = bpAK^aL^{b-1} - \ell.$$
The critical points of the function $P(K, L)$ are solutions (K, L) of the simultaneous equations:
$$\frac{k}{a} = pAK^{a-1}L^b,$$
$$\frac{\ell}{b} = pAK^aL^{b-1}.$$
Multiplying the first equation by K and the second by L, we get
$$\frac{kK}{a} = \frac{\ell L}{b},$$
and so
$$K = \frac{\ell a}{kb}L.$$
Substituting for K in the equation $k/a = pAK^{a-1}L^b$, we get:
$$\frac{k}{a} = pA\left(\frac{\ell a}{kb}\right)^{a-1}L^{a-1}L^b.$$
We must therefore have
$$L^{1-a-b} = pA\left(\frac{a}{k}\right)^a\left(\frac{\ell}{b}\right)^{a-1}.$$
Hence, if $a + b \neq 1$,
$$L = \left[pA\left(\frac{a}{k}\right)^a\left(\frac{\ell}{b}\right)^{(a-1)}\right]^{1/(1-a-b)},$$
and
$$K = \frac{\ell a}{kb}L = \frac{\ell a}{kb}\left[pA\left(\frac{a}{k}\right)^a\left(\frac{\ell}{b}\right)^{(a-1)}\right]^{1/(1-a-b)}.$$

Then, $C = e^{5.20} = 181.3$ and so

$$P(t) = 181.3e^{0.0114t}.$$

In 1990, we have $t = 30$ and the predicted population in millions is

$$P(30) = 181.3e^{0.01141(30)} = 255.3.$$

(b) The difference between the actual and the predicted population is about 6 million or $2\frac{1}{2}\%$. Given that only three data points were used to calculate a and c, this discrepancy is not surprising. Thus, the 1990 census, data does not mean that the assumption of exponential growth is unjustified.

(c) In 2010, we have $t = 50$ and $P(50) = 320.7$.

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We have

$$f_x = 2x(y+1)^3 = 0 \quad \text{only when } x = 0 \text{ or } y = -1$$
$$f_y = 3x^2(y+1)^2 + 2y = 0 \quad \text{never when } y = -1 \text{ and only for } y = 0 \text{ when } x = 0$$

We conclude that $f_x = 0$ and $f_y = 0$ only when $x = 0, y = 0$, so f has only one critical point, namely $(0, 0)$.

The second derivative test at $(0, 0)$ gives

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 2(y+1)^3(6x^2(y+1)+2) - (6x(y+1)^2)^2$$
$$= 2(1)(2) - 0 > 0 \quad \text{when } x = 0, y = 0$$

Since $f_{xx} > 0$ at $(0, 0)$, this means f has a local minimum at $(0, 0)$.

[Alternatively, if we expand $(y+1)^3$, then we can view $f(x, y)$ as $x^2 + y^2 +$ (terms of degree 3 or greater in x and y), which means that f behaves likes $x^2 + y^2$ near $(0, 0)$.]

Although $(0, 0)$ is a local minimum, it cannot be a global minimum since for fixed x, say $x = 1$, the function $f(x, y)$ is a cubic polynomial in y and cubics take on arbitrarily large positive and negative values.

In the single-variable case, suppose a function f defined on the real line is differentiable and its derivative is continuous. Then if f has only one critical point, say $x = 0$, then if that critical point is a local minimum, it must also be a global minimum. This is because f' cannot change sign without $f' = 0$ so we must have $f' < 0$ for $x < 0$ and $f' > 0$ for $x > 0$. Thus f is decreasing for all $x < 0$ and increasing for all $x > 0$, which makes $x = 0$ the global minimum for f.

28. **(a)** We have $f(2, 1) = 120$.

(i) If $x > 20$ then $f(x, y) > 10x > 200 > f(2, 1)$.

(ii) If $y > 20$ then $f(x, y) > 20y > 400 > f(2, 1)$.

(iii) If $x < 0.01$ and $y \le 20$ then $f(x, y) > 80/(xy) > 80/((0.01)(20)) = 400 > f(2, 1)$.

(iv) If $y < 0.01$ and $x \le 20$ then $f(x, y) > 80/(xy) > 80/((20)(0.01)) = 400 > f(2, 1)$.

(b) The continuous function f must achieve a minimum at some point (x_0, y_0) in the closed and bounded region R' : $0.01 \le x \le 20, 0.01 \le y \le 20$. Since $(2, 1)$ is in R', we must have $f(x_0, y_0) \le f(2, 1)$. By part (a), $f(x_0, y_0)$ is less than all values of f in the part of R that is outside R', so $f(x_0, y_0)$ is a minimum for f on all of R. Since (x_0, y_0) is not on the boundary of R, it must be a critical point of f.

(c) The only critical point of f in R is the point $(2, 1)$, so by part (b) f has a global minimum there.

29. **(a)** The function f is continuous in the region R, but R is not closed and bounded so a special analysis is required.

Notice that $f(x, y)$ tends to ∞ as (x, y) tends farther and farther from the origin or tends toward any point on the x or y axis. This suggests that a minimum for f, if it exists, can not be too far from the origin or too close to the axes. For example, if $x > 10$ then $f(x, y) > 4x > 40$, and if $y > 10$ then $f(x, y) > 5y > 50$. If $0 < x < 0.1$ then $f(x, y) > 2/x > 20$, and if $0 < y < 0.1$ then $f(x, y) > 3/y > 30$.

Since $f(1, 1) = 14$, a global minimum for f if it exists must be in the smaller region R' : $0.1 \le x \le 10$, $0.1 \le y \le 10$. The region R' is closed and bounded and so f does have a minimum value at some point in R', and since that value is at most 14, it is also a global minimum for all of R.

(b) Since the region R has no boundary, the minimum value must occur at a critical point of f. At a critical point we have

$$f_x = -\frac{2}{x^2} + 4 = 0 \qquad f_y = -\frac{3}{y^2} + 5 = 0.$$

The only critical point is $(\sqrt{1/2}, \sqrt{3/5}) \approx (0.7071, 0.7746)$, at which f achieves the minimum value $f(\sqrt{1/2}, \sqrt{3/5}) = 4\sqrt{2} + 2\sqrt{15} \approx 13.403$.