

Homework 12 Solutions15.3/3

(Note that the constraint curve is an ellipse, which is a closed and bounded set. f is continuous, so by Theorem 15.1, f has a global max. and a global min. on the curve.)

$$\text{solve } \text{grad } f = \lambda \text{ grad } g, \quad g(x, y) = c$$

$$\Rightarrow \textcircled{1} \quad y = \lambda(8x), \quad \textcircled{2} \quad x = \lambda(2y), \quad \textcircled{3} \quad 4x^2 + y^2 = 8$$

First note that if $x=0$, then $\textcircled{1} \Rightarrow y=0$, and then $\textcircled{3}$ is not correct. Similarly, if $y=0$, $\textcircled{2} \Rightarrow x=0$, and again $\textcircled{3}$ is not correct. So we know x and y are not zero.

$$\textcircled{1} \Rightarrow \lambda = \frac{y}{8x} \quad \text{and} \quad \textcircled{2} \Rightarrow \lambda = \frac{x}{2y}$$

$$\text{so } \frac{y}{8x} = \frac{x}{2y} \Rightarrow 2y^2 = 8x^2 \Rightarrow y = 2x \text{ or } y = -2x$$

In either case, $\textcircled{3} \Rightarrow x^2 = 1 \Rightarrow x = 1 \text{ or } x = -1$.

so the four solutions are $(1, 2), (-1, 2), (1, -2), (-1, -2)$.

These are the points where a contour line of f is tangent to the constraint curve.

We have

$$f(1, 2) = 2, \quad f(-1, 2) = -2, \quad f(1, -2) = -2, \quad f(-1, -2) = 2$$

The constrained global max. is 2, and it occurs at $(1, 2)$ and $(-1, -2)$.

The constrained global min. is -2, and it occurs at $(-1, 2)$ and $(1, -2)$.

15.3/13

$$f(x, y) = xy$$

$$x^2 + 2y^2 \leq 1$$

First consider the unconstrained problem. Find the critical points of f .

$\text{grad } f = \vec{0} \Rightarrow y = 0$ and $x = 0$, so $(0, 0)$ is the only critical point of f . Now $f_{xx}(x, y) = 0$, $f_{yy}(x, y) = 0$, and $f_{xy}(x, y) = 1$, so at $(0, 0)$,

$$D = (0)(0) - (1)^2 = -1 < 0,$$

so $(0, 0)$ is a saddle point. Thus there cannot be a maximum or minimum in the interior of the region $x^2 + 2y^2 \leq 1$. The max and min. must occur on the boundary.

We now tackle the equality constraint. Let $g(x, y) = x^2 + 2y^2$. The equation $\text{grad } f = \lambda \text{ grad } g$ gives

- ① $y = \lambda(2x)$
- ② $x = \lambda(4y)$
- ③ $x^2 + 2y^2 = 1$

Putting ② into ① yields $y = 8\lambda^2 y$, which means

$$y = 0 \text{ or } \lambda = \pm \frac{1}{\sqrt{8}} = \pm \frac{1}{2\sqrt{2}}$$

If $y = 0$, then ② implies $x = 0$, but $x = 0, y = 0$ cannot satisfy ③.

If $\lambda = \frac{1}{2\sqrt{2}}$, put ② into ③ to get $2y^2 + 2y^2 = 1 \Rightarrow y^2 = \frac{1}{4} \Rightarrow y = \pm \frac{1}{2}$.

Then ② gives us the points $(\frac{1}{\sqrt{2}}, \frac{1}{2})$ and $(-\frac{1}{\sqrt{2}}, \frac{1}{2})$

If $\lambda = -\frac{1}{2\sqrt{2}}$, the same steps gives the points $(\frac{1}{\sqrt{2}}, -\frac{1}{2})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{2})$.

(continued \rightarrow)

so the constrained critical points are

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\right) \text{ and } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$$

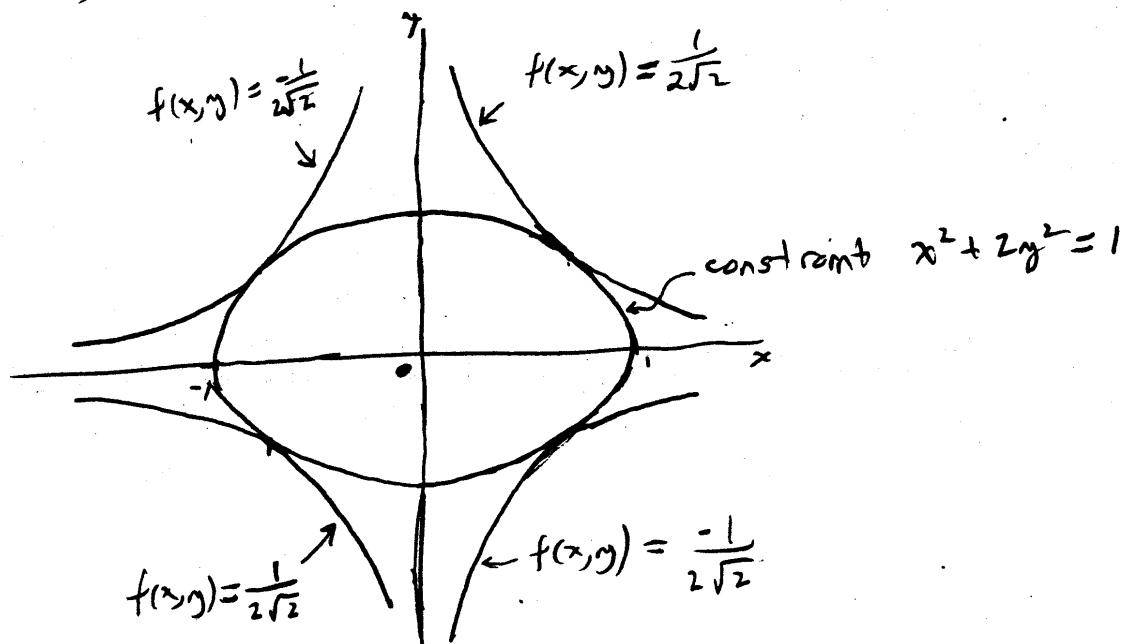
We have

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = \frac{1}{2\sqrt{2}}, \quad f\left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\right) = -\frac{1}{2\sqrt{2}}$$

$$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = -\frac{1}{2\sqrt{2}}, \quad f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right) = \frac{1}{2\sqrt{2}}$$

The maximum value of f is $\frac{1}{2\sqrt{2}}$, and it occurs at $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$.

The minimum value of f is $-\frac{1}{2\sqrt{2}}$, and it occurs at $\left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$.



15.3/15

$$f(x, y) = x^3 + y,$$

$$x + y \geq 1$$

$$g(x, y)$$

The constraint is an inequality, so we do this in two parts.

First consider the unconstrained problem. The first step here is to find the critical points of f . Now

$$\text{grad } f(x, y) = 3x^2 \vec{i} + \vec{j}, \text{ and } \text{grad } f = \vec{0} \Rightarrow$$

$3x^2 = 0$ and $1 = 0$, but this is not possible, so f has no unconstrained critical points.

Now consider the equality constraint on the boundary,

where $x + y = 1$. Let $g(x, y) = x + y$. We solve

$$\text{grad } f = \lambda \text{ grad } g, \text{ or}$$

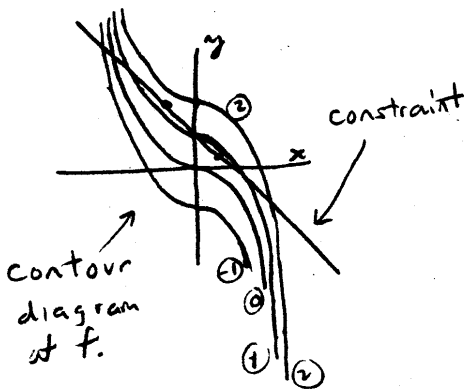
$$\textcircled{1} \quad 3x^2 = \lambda$$

$$\textcircled{2} \quad 1 = \lambda$$

$$\textcircled{3} \quad x + y = 1$$

$\textcircled{1}$ and $\textcircled{2}$ give $x = \pm \frac{1}{\sqrt{3}}$. If $x = \frac{1}{\sqrt{3}}$, $\textcircled{3}$ gives $y = 1 - \frac{1}{\sqrt{3}}$.

If $x = -\frac{1}{\sqrt{3}}$, $\textcircled{3}$ gives $y = 1 + \frac{1}{\sqrt{3}}$. Thus the constrained critical points are $(\frac{1}{\sqrt{3}}, 1 - \frac{1}{\sqrt{3}})$ and $(-\frac{1}{\sqrt{3}}, 1 + \frac{1}{\sqrt{3}})$.



From the contour diagram, we see that there is a constrained local max. at $(\frac{1}{\sqrt{3}}, 1 - \frac{1}{\sqrt{3}})$, and a constrained local min. at $(-\frac{1}{\sqrt{3}}, 1 + \frac{1}{\sqrt{3}})$. However, on the constraint curve, $f(x, y) \rightarrow -\infty$ as $x \rightarrow -\infty$, and $f(x, y) \rightarrow \infty$ as $x \rightarrow \infty$, so the constrained problem has no max and no min.

Therefore, the original full problem has no max. and no min.

15.3/19

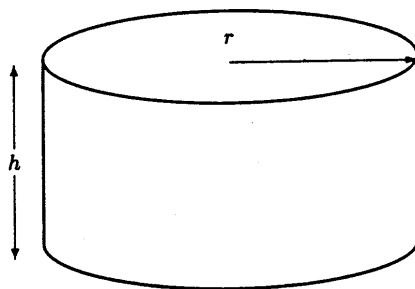


Figure 15.24

Let V be the volume and S be the surface area of the container. Then

$$V = \pi r^2 h \quad \text{and} \quad S = 2\pi r h + 2\pi r^2$$

where h is the height and r is the radius as shown in Figure 15.24. We have $V = 100 \text{ cm}^3$ as our constraint. Since

$$\nabla S = (2\pi h + 4\pi r)\vec{i} + 2\pi r\vec{j} = \pi((2h + 4r)\vec{i} + 2r\vec{j})$$

and
$$\nabla V = 2\pi r h\vec{i} + \pi r^2\vec{j} = \pi(2r h\vec{i} + r^2\vec{j}),$$

at the optimum

$$\nabla S = \lambda \nabla V, \text{ we have}$$

$$\pi((2h + 4r)\vec{i} + 2r\vec{j}) = \pi\lambda(2r h\vec{i} + r^2\vec{j}),$$

that is
$$2h + 4r = 2\lambda r h \quad \text{and} \quad 2r = \lambda r^2, \quad \text{hence} \quad \lambda = \frac{2}{r}.$$

We assume $r \neq 0$ or else we have a very awkward cylinder. Then, plug $\lambda = 2/r$ into the first equation to obtain:

$$2h + 4r = 2\left(\frac{2}{r}\right) r h$$

$$2h + 4r = 4h$$

$$h = 2r.$$

Finally, solve for r and h using the constraint:

$$V = \pi r^2 h = 100$$

$$\pi r^2 (2r) = 100$$

$$r^3 = \frac{50}{\pi}$$

$$r = \sqrt[3]{\frac{50}{\pi}}.$$

Solving for h , we obtain $h = 2r = 2\sqrt[3]{\frac{50}{\pi}}$.

25. Constraint is $G = P_1 x + P_2 y - K = 0$.

Since $\nabla Q = \lambda \nabla G$, we have

$$cax^{a-1}y^b = \lambda P_1 \quad \text{and} \quad cbx^a y^{b-1} = \lambda P_2.$$

Dividing the two equations yields $\frac{cax^{a-1}y^b}{cbx^a y^{b-1}} = \frac{\lambda P_1}{\lambda P_2}$, or simplifying, $\frac{ay}{bx} = \frac{P_1}{P_2}$. Hence, $y = \frac{bP_1}{aP_2}x$.

Substitute into the constraint to obtain $P_1 x + P_2 \frac{bP_1}{aP_2} x = P_1 \left(\frac{a+b}{a}\right) x = K$, giving

$$x = \frac{aK}{(a+b)P_1} \quad \text{and} \quad y = \frac{bK}{(a+b)P_2}.$$

We now check that this is indeed the maximization point. Since $x, y \geq 0$, possible maximization points are $(0, \frac{K}{P_2})$, $(\frac{K}{P_1}, 0)$, and $(\frac{aK}{(a+b)P_1}, \frac{bK}{(a+b)P_2})$. Since $Q = 0$ for the first two points and Q is positive for the last point, it follows that $(\frac{aK}{(a+b)P_1}, \frac{bK}{(a+b)P_2})$ gives the maximal value.

We know that a maximum or minimum value of f subject to the constraint equation $g(x, y) = c$ occurs where $\text{grad } f$ is parallel to $\text{grad } g$, or at the endpoints of the constraint. The vectors $\text{grad } f$ and $\text{grad } g$ are parallel where the graph of $g(x, y) = c$ is tangent to the contours of f , which occurs at approximately $x = 6$ and $y = 6$. At the point $(6, 6)$, we have $f = 400$. The graph of $g(x, y) = c$ crosses the contours $f = 300$, $f = 200$, $f = 100$ but does not cross any contours with f -values greater than 400. We see that the maximum of f subject to the constraint is 400 at the point $(6, 6)$. It appears that f takes on its minimum value (less than 100) at one of the endpoints, which are approximately $(10.5, 0)$ and $(0, 13.5)$.

- 15.3/21
- (a) The company wishes to maximize $P(x, y)$ given the constraint $C(x, y) = 50,000$. The objective function is $P(x, y)$ and the constraint equation is $C(x, y) = 50,000$. The Lagrange multiplier λ is approximately equal to the change in $P(x, y)$ given a one unit increase in the budget constraint. In other words, if we increase the budget by \$1, we can produce about λ more units of the good.
- (b) The company wishes to minimize $C(x, y)$ given the constraint equation $P(x, y) = 2000$. The objective function is $C(x, y)$ and the constraint equation is $P(x, y) = 2000$. The Lagrange multiplier λ is approximately equal to the change in $C(x, y)$ given a one unit increase in the production constraint. In other words, it costs about λ dollars to produce one more unit of the good.

15.3/22 The company wants to maximize $f(x, y) = 500x^{0.6}y^{0.3}$ given the constraint $g(x, y) = 10x + 25y = 2000$. Setting $\text{grad } f = \lambda \text{ grad } g$ gives

$$\begin{aligned} 500(0.6x^{-0.4})y^{0.3} &= 10\lambda, \\ 500x^{0.6}(0.3y^{-0.7}) &= 25\lambda. \end{aligned}$$

From the first equation we have $\lambda = 30y^{0.3}/x^{0.4}$, and from the second equation we have $\lambda = 6x^{0.6}/y^{0.7}$. Setting these equal gives

$$y = 0.2x.$$

Substituting this into the constraint equation $10x + 25y = 2000$ gives $x = 133.33$. Since $y = 0.2x$, the maximum value occurs at $x = 133.33$ and $y = 26.67$.

- (a) The company should purchase 133.33 units of chemical X and 26.67 units of chemical Y. With these purchases, the company will be able to produce $f(133.33, 26.67) = 25,219$ units of chemical Z.
- (b) When $x = 133.33$ and $y = 26.67$, we see that $\lambda = 11.348$. If \$1 is added to the budget, the company will be able to produce about 11.348 additional units of chemical Z.

(a) Let c be the cost of producing the product. Then $c = 10W + 20K = 3000$. At optimum production,

$$\nabla q = \lambda \nabla c.$$

$$\nabla q = \left(\frac{9}{2}W^{-\frac{1}{2}}K^{\frac{1}{4}}\right)\vec{i} + \left(\frac{3}{2}W^{\frac{3}{4}}K^{-\frac{3}{4}}\right)\vec{j}, \text{ and } \nabla c = 10\vec{i} + 20\vec{j}. \text{ Equating we get}$$

$$\frac{9}{2}W^{-\frac{1}{2}}K^{\frac{1}{4}} = \lambda 10, \quad \text{and} \quad \frac{3}{2}W^{\frac{3}{4}}K^{-\frac{3}{4}} = \lambda 20.$$

Dividing yields $K = \frac{1}{6}W$, so substituting into c gives

$$10W - 20\left(\frac{1}{6}W\right) = \frac{40}{3}W = 3000.$$

16.2/11

$$\int_0^3 \int_0^4 (4x+3y) dx dy = \int_0^3 2x^2 + 3xy \Big|_{x=0}^{x=4} dy$$

$$= \int_0^3 (32 + 12y - 0) dy$$

$$= 32y + 6y^2 \Big|_0^3 = 96 + 54 - 0 = 150$$

16.2/7

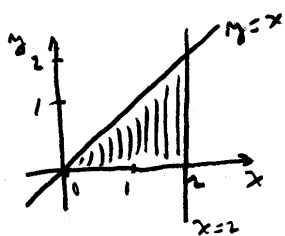
The diagonal line is $3x+4y=1$ or $y = -\frac{3}{4}x + \frac{1}{4}$.

$$\text{or } x = -\frac{4}{3}y + \frac{1}{3}$$

$$\int_R f dA = \int_{x=-1}^3 \int_{y=-2}^{-\frac{3}{4}x + \frac{1}{4}} f(x,y) dy dx$$

$$\text{OR} \int_{y=-2}^1 \int_{x=-1}^{-\frac{4}{3}y + \frac{1}{3}} f(x,y) dx dy$$

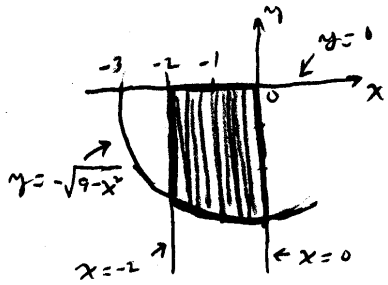
16.2/12



$$\int_{x=0}^2 \int_{y=0}^{y=x} e^{x^2} dy dx = \int_0^2 e^{x^2} x dx = \frac{1}{2} \int_0^2 e^{x^2} 2x dx \quad \left(\begin{array}{l} u=x^2 \\ du=2x dx \end{array} \right)$$

$$= \frac{1}{2} \int_{u=0}^{u=4} e^u du = \frac{1}{2} e^u \Big|_0^4 = \frac{1}{2} (e^4 - 1)$$

16.2/16



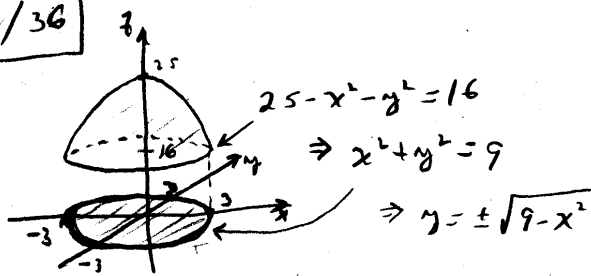
$$\int_{x=-2}^{x=0} \int_{y=-\sqrt{9-x^2}}^{y=0} 2xy \, dy \, dx = \int_{x=-2}^{x=0} xy^2 \Big|_{y=-\sqrt{9-x^2}}^{y=0} dx$$

$$= \int_{x=-2}^{x=0} 0 - x(9-x^2) dx$$

$$= \int_{-2}^0 x^3 - 9x \, dx = \frac{x^4}{4} - \frac{9}{2}x^2 \Big|_{-2}^0$$

$$= 0 - (4 - 18) = 14$$

16.2/36



$$V = \int_{x=-3}^{x=3} \int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} (25 - x^2 - y^2) - (16) \, dy \, dx$$

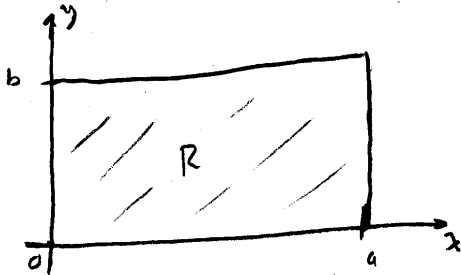
$$= \int_{x=-3}^{x=3} \int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} (9 - x^2 - y^2) \, dy \, dx$$

16.2/38

$$\text{Volume} = \int_0^2 \int_0^2 xy \, dy \, dx = \int_0^2 \frac{1}{2}xy^2 \Big|_{y=0}^{y=2} dx = \int_0^2 2x \, dx = x^2 \Big|_0^2 = 4$$

16.2/49

Pressure is force per unit area, so we can interpret pressure as a "force density". To find the total force, we integrate the pressure.



We have $p(x, y) = k(x^2 + y^2)$ for some constant of proportionality k .

$$\text{Force} = \int_R p dA = \int_{x=0}^{x=a} \int_{y=0}^{y=b} k(x^2 + y^2) dy dx$$

$$= k \int_0^a \left. x^2 y + \frac{y^3}{3} \right|_{y=0}^{y=b} dx = k \int_0^a \left(bx^2 + \frac{b^3}{3} \right) dx$$

$$= k \left(\left[\frac{bx^3}{3} + \frac{b^3}{3}x \right]_0^a \right) = k \left(\frac{ba^3}{3} + \frac{b^3 a}{3} \right)$$

$$= \frac{k}{3} (a^3 b + b^3 a)$$

16.3/4

$$\int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \sin x \cos(y+z) dx dy dz$$

$$= \int_0^{\pi} \int_0^{\pi} -\cos(x) \cos(y+z) \Big|_{x=0}^{x=\pi} dy dz$$

$$= \int_0^{\pi} \int_0^{\pi} 2 \cos(y+z) dy dz = 2 \int_0^{\pi} \sin(y+z) \Big|_{y=0}^{y=\pi} dz$$

$$= 2 \int_0^{\pi} \sin(\pi+z) - \sin(z) dz$$

$$= 2 \left(-\cos(\pi+z) + \cos(z) \right) \Big|_0^{\pi}$$

$$= 2(-1-1-(1+1))$$

$$= -8$$

16.3/10

$$x=1$$

$$z = \sqrt{1-x^2}$$

$$y = \sqrt{1-x^2-y^2}$$

$$\int$$

$$\int$$

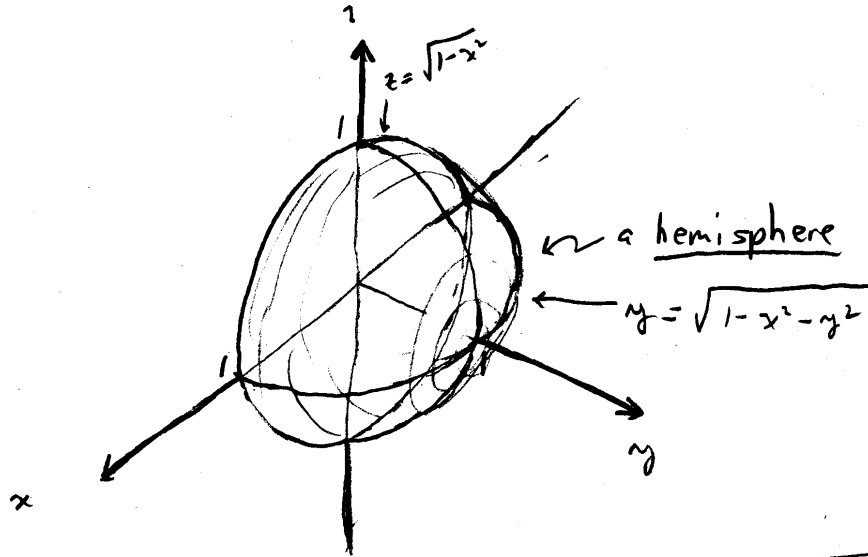
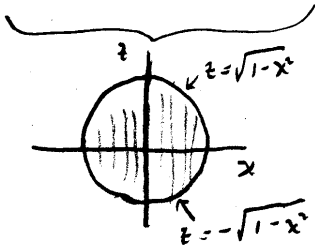
$$\int$$

$$f(x, y, z) \, dy \, dx \, dz$$

$$x=-1$$

$$z = -\sqrt{1-x^2}$$

$$y=0$$



16.3/8

$$x=1 \quad z=1$$

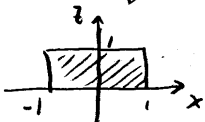
$$y = \sqrt{1-z^2}$$

$$\int \int$$

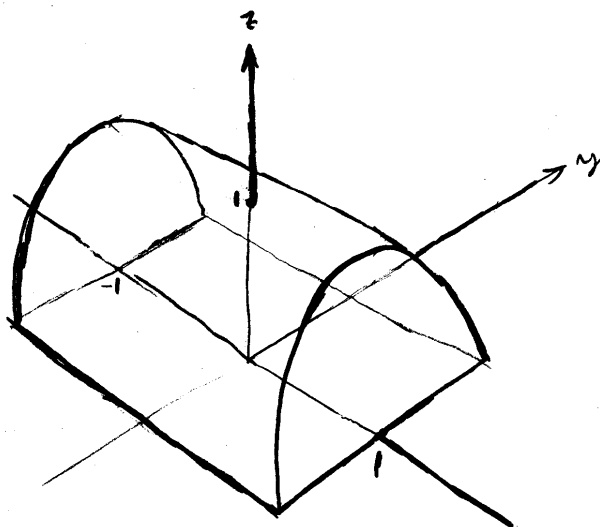
$$f(x, y, z) \, dy \, dz \, dx$$

$$x=-1 \quad z=0$$

$$y = -\sqrt{1-z^2}$$



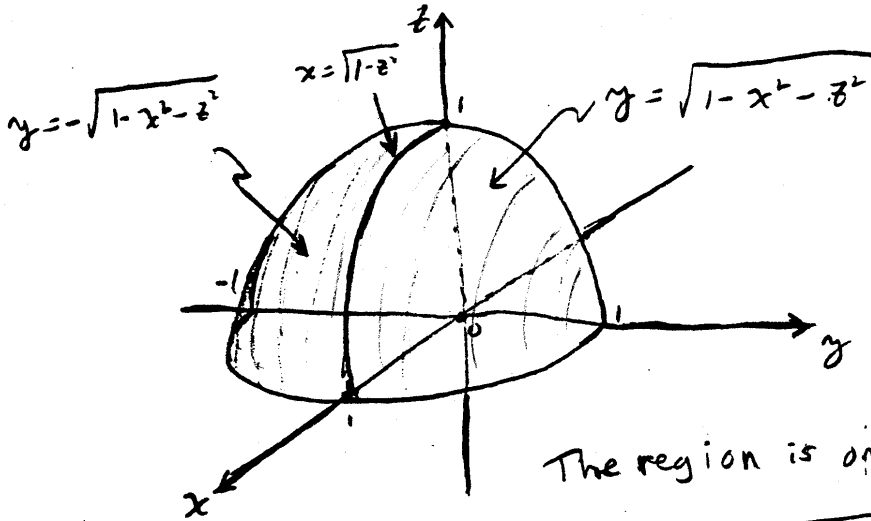
Note: $y = \pm \sqrt{1-z^2} \Rightarrow y^2 + z^2 = 1$
 \Rightarrow cylinder around the x axis



This is the upper half ($z \geq 0$)
of a cylinder (radius 1)
around the x axis.

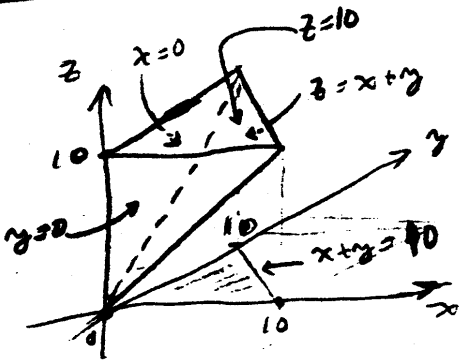
16.3/12

$$\int_{z=0}^{z=1} \int_{x=0}^{x=\sqrt{1-z^2}} \int_{y=-\sqrt{1-x^2-z^2}}^{y=\sqrt{1-x^2-z^2}} f(x, y, z) \, dy \, dx \, dz$$



The region is one quarter of a sphere.

16.3/18



$$V = \int_{x=0}^{x=10} \int_{y=0}^{y=10-x} \int_{z=x+y}^{z=10} dz \, dy \, dx$$

$$= \int_0^{10} \int_0^{10-x} (10-x-y) \, dy \, dx$$

$$= \int_0^{10} \left((10-x)y - \frac{y^2}{2} \right) \Big|_{y=0}^{y=10-x} dx$$

$$= \int_0^{10} (10-x)^2 - \frac{(10-x)^2}{2} \, dx = \frac{1}{2} \int_0^{10} (10-x)^2 \, dx$$

$$= -\frac{1}{2} \frac{(10-x)^3}{3} \Big|_0^{10} = \frac{1000}{6} = \frac{500}{3}$$