Math 113 Fall '05

Homework 12 Solutions

15.3/3

(Note that the constraint curve is an ellipse, which is a closed and bounded set. f is continuous, so by Theorem 15.1, f has a global max. and a global min. on the curve.)

Solve $\operatorname{grad} f = \lambda \operatorname{grad} g$, $g(x,y)=c$

$$\Rightarrow \text{①}\ y = \lambda(8x), \quad \text{②}\ x = \lambda(2y), \quad \text{③}\ 4x^2+y^2=8$$

First note that if $x=0$, then ① $\Rightarrow y=0$, and then ③ is not correct. Similarly, if $y=0$, ② $\Rightarrow x=0$, and again ③ is not correct. So we know x and y are not zero.

① $\Rightarrow \lambda = \dfrac{y}{8x}$ and ② $\Rightarrow \lambda = \dfrac{x}{2y}$

so $\dfrac{y}{8x} = \dfrac{x}{2y} \Rightarrow 2y^2 = 8x^2 \Rightarrow y = 2x$ OR $y=-2x$

In either case, ③ $\Rightarrow x^2 = 1 \Rightarrow x = 1$ or $x=-1$.

so the four solutions are $(1,2)$, $(-1,2)$, $(1,-2)$, $(-1,-2)$.

These are the points where a contour line of f is tangent to the constraint curve.

We have

$$f(1,2) = 2, \quad f(-1,2) = -2, \quad f(1,-2) = -2, \quad f(-1,-2) = 2$$

The constrained global max. is 2, and it occurs at $(1,2)$ and $(-1,-2)$.

The constrained global min. is -2, and it occurs at $(-1,2)$ and $(1,-2)$.

$\boxed{15.3/13}$ $f(x,y) = xy$ $x^2 + 2y^2 \le 1$

First consider the unconstrained problem. Find the critical points of f.

$\operatorname{grad} f = \vec{0} \Rightarrow y = 0$ and $x = 0$, so $(0,0)$ is the only critical point of f. Now $f_{xx}(x,y) = 0$, $f_{yy}(x,y) = 0$, and $f_{xy}(x,y) = 1$, so at $(0,0)$,

$$D = (0)(0) - (1)^2 = -1 < 0,$$

so $(0,0)$ is a saddle point. Thus there cannot be a maximum or minimum in the interior of the region $x^2+2y^2 \le 1$. The max and min. must occur on the boundary.

We now tackle the equality constraint. Let $g(x,y) = x^2+2y^2$. The equation $\operatorname{grad} f = \lambda \operatorname{grad} g$ gives

① $y = \lambda(2x)$

② $x = \lambda(4y)$

③ $x^2 + 2y^2 = 1$

Putting ② into ① yields $y = 8\lambda^2 y$, which means

$y = 0$ or $\lambda = \pm\frac{1}{\sqrt{8}} = \pm\frac{1}{2\sqrt{2}}$

If $y = 0$, then ② implies $x = 0$, but $x=0, y=0$ cannot satisfy ③.

If $\lambda = \frac{1}{2\sqrt{2}}$, put ② into ③ to get $2y^2 + 2y^2 = 1 \Rightarrow y^2 = \frac{1}{4} \Rightarrow y = \pm\frac{1}{2}$.

Then ② gives us the points $(\frac{1}{\sqrt{2}}, \frac{1}{2})$ and $(-\frac{1}{\sqrt{2}}, \frac{1}{2})$

If $\lambda = -\frac{1}{2\sqrt{2}}$, the same steps gives the points $(\frac{1}{\sqrt{2}}, -\frac{1}{2})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{2})$.

(continued →)

so the constrained critical points are

$$\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{2}\right),\ \left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{2}\right),\ \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{2}\right) \text{ and } \left(-\tfrac{1}{\sqrt{2}}, -\tfrac{1}{2}\right)$$

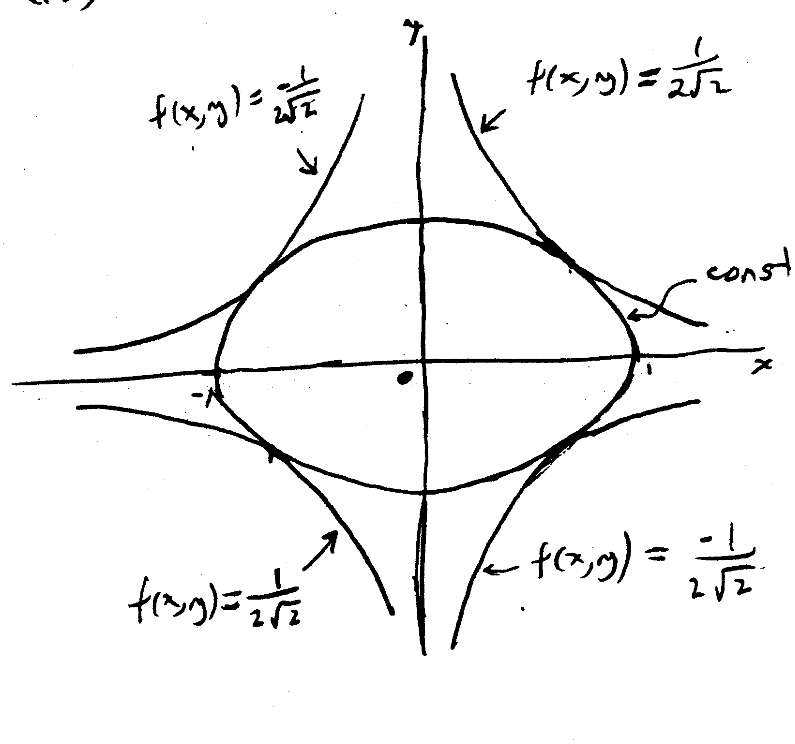
We have

$$f\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{2}\right) = \tfrac{1}{2\sqrt{2}}, \quad f\left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{2}\right) = -\tfrac{1}{2\sqrt{2}}$$

$$f\left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{2}\right) = -\tfrac{1}{2\sqrt{2}}, \quad f\left(\tfrac{-1}{\sqrt{2}}, -\tfrac{1}{2}\right) = \tfrac{1}{2\sqrt{2}}.$$

The maximum value of f is $\frac{1}{2\sqrt{2}}$, and it occurs at $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$.

The minimum value of f is $-\frac{1}{2\sqrt{2}}$, and it occurs at $\left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$.



$\boxed{15.3/15}$ $f(x,y) = x^3 + y$, $\underbrace{x+y}_{g(x,y)} \geq 1$

The constraint is an <u>inequality</u>, so we do this in two parts. First consider the unconstrained problem. The first step here is to find the critical points of f. Now

$$\text{grad } f(x,y) = 3x^2\vec{i} + \vec{j}, \text{ and } \text{grad } f = \vec{0} \Rightarrow$$

$3x^2 = 0$ and $1 = 0$, but this is not possible, so f has <u>no</u> unconstrained critical points.

Now consider the <u>equality</u> constraint on the boundary, where $x+y=1$. Let $g(x,y) = x+y$. We solve

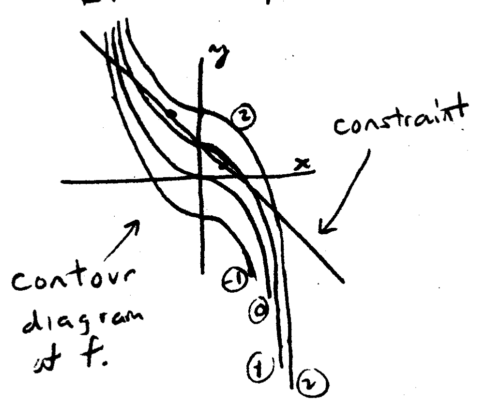
grad $f = \lambda$ grad g, or

① $3x^2 = \lambda$

② $1 = \lambda$

③ $x+y=1$

① and ② give $x = \pm\frac{1}{\sqrt{3}}$. If $x = \frac{1}{\sqrt{3}}$, ③ gives $y = 1 - \frac{1}{\sqrt{3}}$. If $x = -\frac{1}{\sqrt{3}}$, ③ gives $y = 1 + \frac{1}{\sqrt{3}}$. Thus the <u>constrained</u> critical points are $\left(\frac{1}{\sqrt{3}}, 1-\frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, 1+\frac{1}{\sqrt{3}}\right)$.



From the contour diagram, we see that there is a constrained local max. at $\left(-\frac{1}{\sqrt{3}}, 1+\frac{1}{\sqrt{3}}\right)$, and a constrained local min. at $\left(\frac{1}{\sqrt{3}}, 1-\frac{1}{\sqrt{3}}\right)$. However, on the constraint curve, $f(x,y) \to -\infty$ as $x \to -\infty$, and $f(x,y) \to \infty$ as $x \to \infty$, so the constrained problem has no max and no min.

Therefore, the original full problem has <u>no max</u>. and <u>no min</u>.

15.3/19

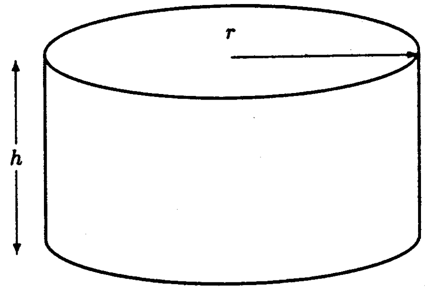


Figure 15.24

Let V be the volume and S be the surface area of the container. Then

$$V = \pi r^2 h \quad \text{and} \quad S = 2\pi r h + 2\pi r^2$$

where h is the height and r is the radius as shown in Figure 15.24. We have $V = 100\ \text{cm}^3$ as our constraint. Since

$$\nabla S = (2\pi h + 4\pi r)\vec{i} + 2\pi r\vec{j} = \pi((2h+4r)\vec{i} + 2r\vec{j})$$
$$\text{and} \qquad \nabla V = 2\pi r h\vec{i} + \pi r^2\vec{j} = \pi(2rh\vec{i} + r^2\vec{j}),$$

at the optimum

$$\nabla S = \lambda \nabla V, \text{ we have}$$
$$\pi((2h+4r)\vec{i} + 2r\vec{j}) = \pi\lambda(2rh\vec{i} + r^2\vec{j}),$$
$$\text{that is} \qquad 2h + 4r = 2\lambda r h \quad \text{and} \quad 2r = \lambda r^2, \qquad \text{hence} \quad \lambda = \frac{2}{r}.$$

We assume $r \neq 0$ or else we have a very awkward cylinder. Then, plug $\lambda = 2/r$ into the first equation to obtain:

$$\begin{aligned} 2h + 4r &= 2\left(\frac{2}{r}\right) r h \\ 2h + 4r &= 4h \\ h &= 2r. \end{aligned}$$

Finally, solve for r and h using the constraint:

$$\begin{aligned} V = \pi r^2 h &= 100 \\ \pi r^2 (2r) &= 100 \\ r^3 &= \frac{50}{\pi} \\ r &= \sqrt[3]{\frac{50}{\pi}}. \end{aligned}$$

Solving for h, we obtain $h = 2r = 2\sqrt[3]{\frac{50}{\pi}}$.

25. Constraint is $G = P_1 x + P_2 y - K = 0$.
Since $\nabla Q = \lambda \nabla G$, we have

$$cax^{a-1}y^b = \lambda P_1 \quad \text{and} \quad cbx^a y^{b-1} = \lambda P_2.$$

Dividing the two equations yields $\dfrac{cax^{a-1}y^b}{cbx^a y^{b-1}} = \dfrac{\lambda P_1}{\lambda P_2}$, or simplifying, $\dfrac{ay}{bx} = \dfrac{P_1}{P_2}$. Hence, $y = \dfrac{bP_1}{aP_2}x$.

Substitute into the constraint to obtain $P_1 x + P_2 \dfrac{bP_1}{aP_2} x = P_1\left(\dfrac{a+b}{a}\right) x = K$, giving

$$x = \frac{aK}{(a+b)P_1} \quad \text{and} \quad y = \frac{bK}{(a+b)P_2}.$$

We now check that this is indeed the maximization point. Since $x, y \geq 0$, possible maximization points are $(0, \dfrac{K}{P_2})$, $(\dfrac{K}{P_1}, 0)$, and $(\dfrac{aK}{(a+b)P_1}, \dfrac{bK}{(a+b)P_2})$. Since $Q = 0$ for the first two points and Q is positive for the last point, it follows that $(\dfrac{aK}{(a+b)P_1}, \dfrac{bK}{(a+b)P_2})$ gives the maximal value.

[redacted] We know that a maximum or minimum value of f subject to the constraint equation $g(x,y) = c$ occurs where grad f is parallel to grad g, or at the endpoints of the constraint. The vectors grad f and grad g are parallel where the graph of $g(x,y) = c$ is tangent to the contours of f, which occurs at approximately $x = 6$ and $y = 6$. At the point $(6,6)$, we have $f = 400$. The graph of $g(x,y) = c$ crosses the contours $f = 300$, $f = 200$, $f = 100$ but does not cross any contours with f-values greater than 400. We see that the maximum of f subject to the constraint is 400 at the point $(6,6)$. It appears that f takes on its minimum value (less than 100) at one of the endpoints, which are approximately $(10.5, 0)$ and $(0, 13.5)$.

15.3/21 [redacted] **(a)** The company wishes to maximize $P(x,y)$ given the constraint $C(x,y) = 50,000$. The objective function is $P(x,y)$ and the constraint equation is $C(x,y) = 50,000$. The Lagrange multiplier λ is approximately equal to the change in $P(x,y)$ given a one unit increase in the budget constraint. In other words, if we increase the budget by \$1, we can produce about λ more units of the good.

(b) The company wishes to minimize $C(x,y)$ given the constraint equation $P(x,y) = 2000$. The objective function is $C(x,y)$ and the constraint equation is $P(x,y) = 2000$. The Lagrange multiplier λ is approximately equal to the change in $C(x,y)$ given a one unit increase in the production constraint. In other words, it costs about λ dollars to produce one more unit of the good.

15.3/22 [redacted] The company wants to maximize $f(x,y) = 500x^{0.6}y^{0.3}$ given the constraint $g(x,y) = 10x + 25y = 2000$. Setting grad $f = \lambda$ grad g gives

$$500(0.6x^{-0.4})y^{0.3} = 10\lambda,$$
$$500x^{0.6}(0.3y^{-0.7}) = 25\lambda.$$

From the first equation we have $\lambda = 30y^{0.3}/x^{0.4}$, and from the second equation we have $\lambda = 6x^{0.6}/y^{0.7}$. Setting these equal gives

$$y = 0.2x.$$

Substituting this into the constraint equation $10x + 25y = 2000$ gives $x = 133.33$. Since $y = 0.2x$, the maximum value occurs at $x = 133.33$ and $y = 26.67$.

(a) The company should purchase 133.33 units of chemical X and 26.67 units of chemical Y. With these purchases, the company will be able to produce $f(133.33, 26.67) = 25,219$ units of chemical Z.

(b) When $x = 133.33$ and $y = 26.67$, we see that $\lambda = 11.348$. If \$1 is added to the budget, the company will be able to produce about 11.348 additional units of chemical Z.

[redacted] **(a)** Let c be the cost of producing the product. Then $c = 10W + 20K = 3000$. At optimum production,

$$\nabla q = \lambda \nabla c.$$

$\nabla q = \left(\frac{9}{2}W^{-\frac{1}{4}}K^{\frac{1}{4}}\right)\vec{i} + \left(\frac{3}{2}W^{\frac{3}{4}}K^{-\frac{3}{4}}\right)\vec{j}$. and $\nabla c = 10\vec{i} + 20\vec{j}$. Equating we get

$$\frac{9}{2}W^{-\frac{1}{4}}K^{\frac{1}{4}} = \lambda 10. \quad \text{and} \quad \frac{3}{2}W^{\frac{3}{4}}K^{-\frac{3}{4}} = \lambda 20.$$

Dividing yields $K = \frac{1}{6}W$, so substituting into c gives

$$10W - 20\left(\frac{1}{6}W\right) = \frac{40}{3}W = 3000.$$

$\boxed{16.2/1}$

$$\int_0^3 \int_0^4 (4x+3y)\,dx\,dy = \int_0^3 2x^2+3xy\Big|_{x=0}^{x=4}\,dy$$

$$= \int_0^3 (32+12y-0)\,dy$$

$$= 32y+6y^2\Big|_0^3 = 96+54-0 = 150$$

$\boxed{16.2/7}$ The diagonal line is $3x+4y=1$ or $y=-\frac{3}{4}x+\frac{1}{4}$.

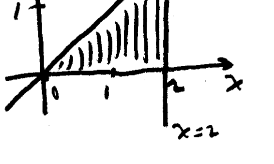
or $x=-\frac{4}{3}y+\frac{1}{3}$

$$\int_R f\,dA = \int_{x=-1}^{3} \int_{y=-2}^{-\frac{3}{4}x+\frac{1}{4}} f(x,y)\,dy\,dx$$

<u>OR</u>

$$\int_{y=-2}^{1} \int_{x=-1}^{-\frac{4}{3}y+\frac{1}{3}} f(x,y)\,dx\,dy$$

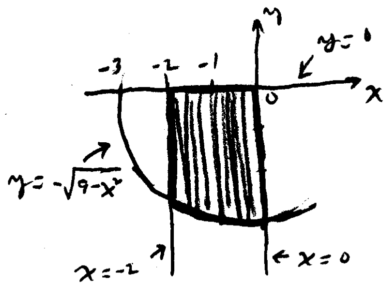
$\boxed{16.2/12}$



$$\int_{x=0}^{x=2} \int_{y=0}^{y=x} e^{x^2}\,dy\,dx = \int_0^2 e^{x^2} x\,dx = \frac{1}{2}\int_0^2 e^{x^2}\,2x\,dx \quad \begin{pmatrix} u=x^2 \\ du=2x\,dx \end{pmatrix}$$

$$= \frac{1}{2}\int_{u=0}^{u=4} e^u\,du = \frac{1}{2}e^u\Big|_0^4 = \frac{1}{2}(e^4-1)$$

$\boxed{16.2/16}$



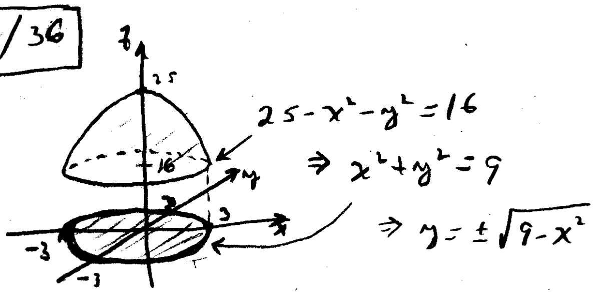
$$\int_{x=-2}^{x=0}\int_{y=-\sqrt{9-x^2}}^{y=0} 2xy\,dy\,dx = \int_{x=-2}^{x=0} xy^2\Big|_{y=-\sqrt{9-x^2}}^{y=0} dx$$

$$= \int_{x=-2}^{x=0} 0 - x(9-x^2)\,dx$$

$$= \int_{-2}^{0} x^3 - 9x\,dx = \frac{x^4}{4} - \frac{9}{2}x^2\Big|_{-2}^{0}$$

$$= 0 - (4-18) = 14$$

$\boxed{16.2/36}$



$$V = \int_{x=-3}^{x=3}\int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} (25-x^2-y^2)-(16)\,dy\,dx$$

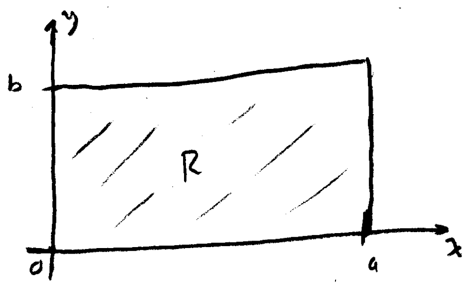
$$= \int_{x=-3}^{x=3}\int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} (9-x^2-y^2)\,dy\,dx$$

$\boxed{16.2/38}$

$$\text{Volume} = \int_0^2\int_0^2 xy\,dy\,dx = \int_0^2 \frac{1}{2}xy^2\Big|_{y=0}^{y=2} dx = \int_0^2 2x\,dx = x^2\Big|_0^2 = 4$$

16.2/49

Pressure is force per unit area, so we can interpret pressure as a "force density". To find the total force, we integrate the pressure.



We have $p(x,y) = k(x^2+y^2)$ for some constant of proportionality k.

$$\text{Force} = \int_R p\,dA = \int_{x=0}^{x=a}\int_{y=0}^{y=b} k(x^2+y^2)\,dy\,dx$$

$$= k\int_0^a x^2y + \frac{y^3}{3}\Big|_{y=0}^{y=b} dx = k\int_0^a\left(bx^2+\frac{b^3}{3}\right)dx$$

$$= k\left(\left[b\frac{x^3}{3}+\frac{b^3}{3}x\right]_0^a\right) = k\left(\frac{ba^3}{3}+\frac{b^3a}{3}\right)$$

$$= \frac{k}{3}\left(a^3b+b^3a\right)$$

$\boxed{16.3/4}$

$$\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \sin x \cos(y+z)\, dx\, dy\, dz$$

$$= \int_0^{\pi}\int_0^{\pi} -\cos(x)\cos(y+z)\Big|_{x=0}^{x=\pi} dy\, dz$$

$$= \int_0^{\pi}\int_0^{\pi} 2\cos(y+z)\, dy\, dz = 2\int_0^{\pi} \sin(y+z)\Big|_{y=0}^{y=\pi} dz$$

$$= 2\int_0^{\pi} \sin(\pi+z) - \sin(z)\, dz$$

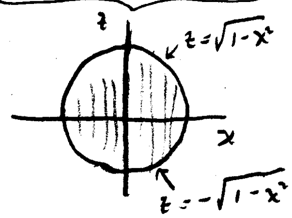
$$= 2\left(-\cos(\pi+z) + \cos(z)\right)\Big|_0^{\pi}$$

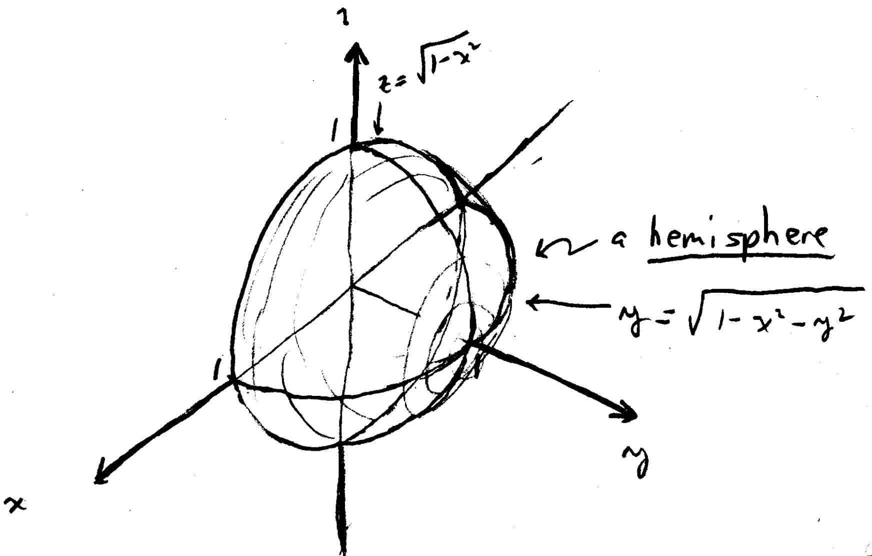
$$= 2(-1-1-(1+1))$$

$$= -8$$

$\boxed{16.3/10}$

$$\int_{x=-1}^{x=1} \int_{z=-\sqrt{1-x^2}}^{z=\sqrt{1-x^2}} \int_{y=0}^{y=\sqrt{1-x^2-y^2}} f(x,y,z)\, dy\, dx\, dz$$

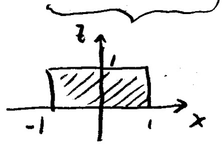


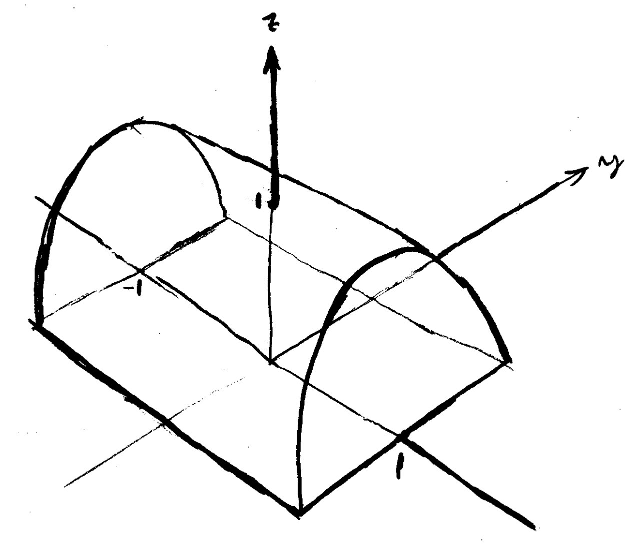


$\boxed{16.3/8}$

$$\int_{x=-1}^{x=1} \int_{z=0}^{z=1} \int_{y=-\sqrt{1-z^2}}^{y=\sqrt{1-z^2}} f(x,y,z)\, dy\, dz\, dx$$

Note: $y=\pm\sqrt{1-z^2} \Rightarrow y^2+z^2=1$
$\Rightarrow$ cylinder around the x axis

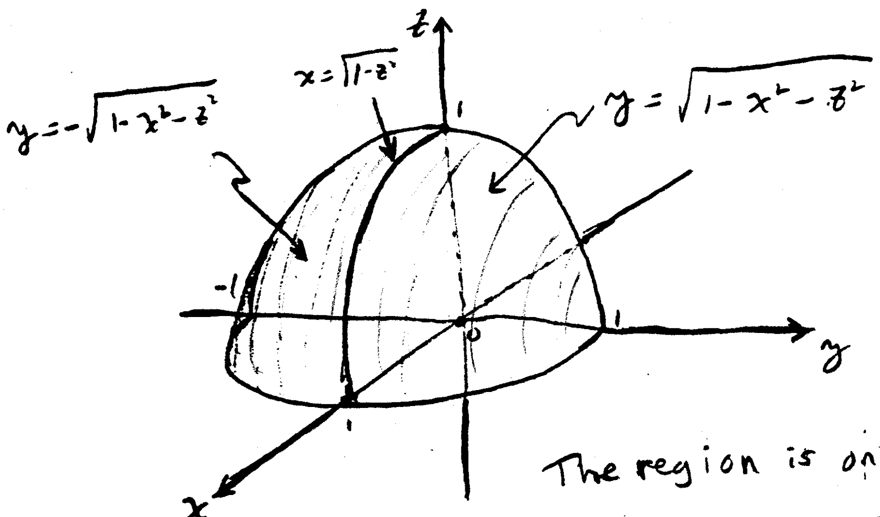




This is the upper half $(z\geq 0)$ of a cylinder (radius 1) around the x axis.

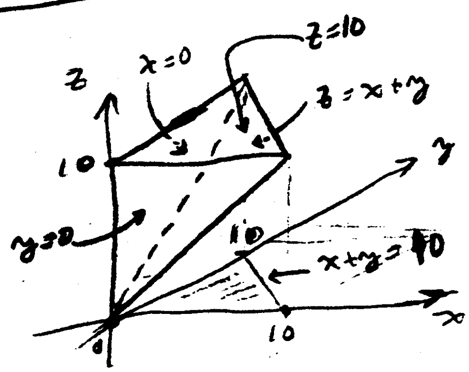
$\boxed{16.3/12}$

$$\int_{z=0}^{z=1}\int_{x=0}^{x=\sqrt{1-z^2}}\int_{y=-\sqrt{1-x^2-z^2}}^{y=\sqrt{1-x^2-z^2}} f(x,y,z)\,dy\,dx\,dz$$



The region is one quarter of a sphere.

$\boxed{16.3/18}$



$$V=\int_{x=0}^{x=10}\int_{y=0}^{y=10-x}\int_{z=x+y}^{z=10} dz\,dy\,dx$$

$$=\int_0^{10}\int_0^{10-x}(10-x-y)\,dy\,dx$$

$$=\int_0^{10}\left((10-x)y-\frac{y^2}{2}\right)\Bigg|_{y=0}^{y=10-x}dx$$

$$=\int_0^{10}(10-x)^2-\frac{(10-x)^2}{2}\,dx=\frac{1}{2}\int_0^{10}(10-x)^2\,dx$$

$$=-\frac{1}{2}\frac{(10-x)^3}{3}\Bigg|_0^{10}=\frac{1000}{6}=\frac{500}{3}$$