1. Suppose the motion of a particle is given by

\[ x = 4 \cos t, \quad y = \sin t. \]

(a) Describe the motion of the particle, and sketch the curve along which the particle travels.

(b) Find the velocity and acceleration vectors of the particle.

(c) Find the times \( t \) and the points on the curve where the speed of the particle is greatest.

(d) Find the times \( t \) and the points on the curve where the magnitude of the acceleration is greatest.

2. Find the coordinates of the points where the line

\[ x = t, \quad y = 1 + t, \quad z = 5t \]

intersects the surface

\[ z = x^2 + y^2. \]

3. Let

\[ g(x, y, z) = e^{-(x+y)^2} + z^2(x + y). \]

Suppose that a piece of fruit is sitting on a table in a room, and at each point \((x, y, z)\) in the space within the room, \(g(x, y, z)\) gives the strength of the odor of the fruit. Furthermore, suppose that a certain bug always flies in the direction in which the fruit odor increases fastest. Suppose also that the bug always flies with a speed of 2 feet/second.

What is the velocity vector of the bug when it is at the position \((2, -2, 1)\)?

4. The path of a particle in space is given by the functions \(x(t) = 2t, \ y(t) = \cos(t), \) and \(z(t) = \sin(t).\) Suppose the temperature in this space is given by a function \(H(x, y, z).\)

Find \( \frac{dH}{dt}, \) the rate of change of the temperature at the particle’s position. (Since the actual function \(H(x, y, z)\) is not given, your answer will be in terms of derivatives of \(H.)\)
5. Let 
\[ f(x, y) = x^3 - xy + \cos(\pi(x + y)). \]

(a) Find a vector normal to the level curve \( f(x, y) = 1 \) at the point where \( x = 1, y = 1 \).

(b) Find the equation of the line tangent to the level curve \( f(x, y) = 1 \) at the point where \( x = 1, y = 1 \).

(c) Find a vector normal to the graph \( z = f(x, y) \) at the point \( x = 1, y = 1 \).

(d) Find the equation of the plane tangent to the graph \( z = f(x, y) \) at the point \( x = 1, y = 1 \).

6. Suppose \( f \) is a differentiable function such that
\[ f(1, 3) = 1, \quad f_x(1, 3) = 2, \quad f_y(1, 3) = 4, \]
\[ f_{xx}(1, 3) = 2, \quad f_{xy}(1, 3) = -1, \quad \text{and} \quad f_{yy}(1, 3) = 4. \]

(a) Find \( \text{grad} f(1, 3) \).

(b) Find a vector in the plane that is perpendicular to the contour line \( f(x, y) = 1 \) at the point \((1, 3)\).

(c) Find a vector that is perpendicular to the surface \( z = f(x, y) \) (i.e. the graph of \( f \)) at the point \((1, 3, 1)\).

(d) At the point \((1, 3)\), what is the rate of change of \( f \) in the direction \( \vec{i} + \vec{j} \)?

(e) Use a quadratic approximation to estimate \( f(1.2, 3.3) \).

7. We say that a line in 3-space is \textit{normal to a surface} at a point of intersection if the line is normal to (i.e. perpendicular to) the tangent plane of the surface at that point.

Let \( S \) be the surface defined by
\[ x^2 + y^2 + 2z^2 = 4. \]

(a) Find the parametric equations of the line that is normal to the surface \( S \) at the point \((1, 1, 1)\).

(b) The line found in (a) will intersect the surface \( S \) at two points. One of them is \((1, 1, 1)\), by construction. Find the other point of intersection.

8. Let 
\[ f(x, y) = (x - y)^3 + 2xy + x^2 - y. \]

(a) Find the function \( L(x, y) \) that gives the linear approximation of \( f \) near the point \((1, 2)\).

(b) Find the function \( Q(x, y) \) that gives the quadratic approximation of \( f \) near the point \((1, 2)\).
9. For each of the following functions, determine the set of points where the function is not differentiable. Briefly explain how you know it is not differentiable; use a picture if it helps.

(You do not have to prove that it is not differentiable; just identify the set of points based on your understanding of what differentiable means.)

(a) \( f(x, y) = |x^2 + y^2 - 1| \)
(b) \( f(x, y) = (x^2 + y^2)^{1/4} \)
(c) \( f(x, y) = e^{-x^2 + y} \)
(d) \( f(x, y) = \frac{x^3 - xy + 1}{x^2 - y^2} \)

10. Suppose \( w = Q(x, y, z) \), where \( Q \) is a differentiable function. Next suppose that \( x = f(t), y = g(t) \) and \( z = h(t) \).

(a) Use the chain rule to find an expression for \( \frac{dw}{dt} \) in terms of \( Q, f, g, h \) and their derivatives (e.g. \( Q_x, f', \) etc.).

(b) Show that the expression in (a) may be written as

\[
\frac{dw}{dt} = (\text{grad} \ Q(\vec{r}(t))) \cdot \frac{d\vec{r}}{dt},
\]

where \( \cdot \) is the dot product, \( \vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} \) is the vector form of the parameterized curve \( x = f(t), y = g(t), \) and \( z = h(t) \), and, if \( \vec{r} = a\vec{i} + b\vec{j} + c\vec{k} \), \( Q(\vec{r}) \) means \( Q(a, b, c) \).
SOLUTIONS

1. (a) Note that \((x/4)^2 + y^2 = \cos^2 t + \sin^2 t = 1\), so the path of the particle is the ellipse \((x/4)^2 + y^2 = 1\). The motion is counter-clockwise around the ellipse.

(b) The path is \(\vec{r}(t) = (4 \cos t)\vec{i} + (\sin t)\vec{j}\), so the velocity is
\[
\vec{v}(t) = \vec{r}'(t) = (-4 \sin t)\vec{i} + (\cos t)\vec{j}
\]
and the acceleration is
\[
\vec{a}(t) = \vec{r}''(t) = (-4 \cos t)\vec{i} + (-\sin t)\vec{j}.
\]

(c) The speed is
\[
v = \|\vec{v}(t)\| = \sqrt{(-4 \sin t)^2 + (\cos t)^2} = \sqrt{16 \sin^2 t + \cos^2 t}.
\]
This has a maximum or minimum when \(dv/dt = 0\). Now
\[
\frac{dv}{dt} = \frac{15 \sin t \cos t}{\sqrt{16 \sin^2 t + \cos^2 t}},
\]
so \(dv/dt = 0\) when \(\sin t = 0\) or \(\cos t = 0\). This means \(t = n\pi\) or \(t = \pi/2 + n\pi\), where \(n\) is any integer. When \(t = n\pi\), \(v = 1\), and when \(t = \pi/2 + n\pi\), \(v = 4\). Thus the maximum speed is \(v = 4\). It occurs when \(t = \pi/2 + n\pi\). When \(n\) is even, the point in the plane where this occurs is \((0, 1)\), and when \(n\) is odd, the point is \((0, -1)\).

(d) The magnitude of the acceleration is
\[
a = \|\vec{a}(t)\| = \sqrt{(-4 \cos t)^2 + (-\sin t)^2} = \sqrt{16 \cos^2 t + \sin^2 t}.
\]
This has a maximum or minimum when \(da/dt = 0\). Now
\[
\frac{da}{dt} = \frac{-15 \sin t \cos t}{\sqrt{16 \cos^2 t + \sin^2 t}},
\]
so \( \frac{da}{dt} = 0 \) when \( \sin t = 0 \) or \( \cos t = 0 \). This means \( t = n\pi \) or \( t = \pi/2 + n\pi \), where \( n \) is any integer. When \( t = n\pi \), \( a = 4 \), and when \( t = \pi/2 + n\pi \), \( a = 1 \). Thus the maximum magnitude of the acceleration is \( a = 4 \). It occurs when \( t = n\pi \). When \( n \) is even, the point in the plane where this occurs is \((0, 4)\), and when \( n \) is odd, the point is \((0, -4)\).

2. We have \( x = t \), \( y = 1 + t \) and \( z = 5t \), and we want \( z = x^2 + y^2 \), so we must solve

\[
5t = t^2 + (1 + t)^2 \implies 2t^2 - 3t + 1 = 0 \implies t = \frac{3 \pm \sqrt{9 - 8}}{4} = \frac{3 \pm 1}{4}.
\]

So the line intersects the surface when \( t = 1/2 \) and when \( t = 1 \). When \( t = 1/2 \), the point of intersection is \((1/2, 3/2, 5/2)\), and when \( t = 1 \), the point of intersection is \((1, 2, 5)\).

3. Since the bug flies in the direction in which the fruit odor increases fastest, it flies in the direction of \( \nabla g \). The gradient of \( g \) is

\[
\nabla g(x, y, z) = \left( -2(x + y)e^{-(x+y)^2} + z^2 \right) \vec{i} + \left( -2(x + y)e^{-(x+y)^2} + z^2 \right) \vec{j} + (2z(x + y)) \vec{k},
\]

and

\[
\nabla g(2, -2, 1) = \vec{i} + \vec{j}.
\]

The bug always has a speed of 2, so the velocity vector must have a magnitude of 2. A vector with magnitude 2 and in the same direction as the gradient is

\[
2 \frac{\nabla g(2, -2, 1)}{\| \nabla g(2, -2, 1) \|} = \frac{2}{\sqrt{2}} (\vec{i} + \vec{j}).
\]

4. By the chain rule,

\[
\frac{dH}{dt} = \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial y} \frac{dy}{dt} + \frac{\partial H}{\partial z} \frac{dz}{dt} = 2 \frac{\partial H}{\partial x} - \sin t \frac{\partial H}{\partial y} + \cos t \frac{\partial H}{\partial z}
\]

5. (a) The gradient of \( f \) is normal to the level curve at each point. We find

\[
\nabla f(x, y) = (3x^2 - y - \pi \sin(\pi(x + y)))\vec{i} + (-x - \pi \sin(\pi(x + y)))\vec{j},
\]

and

\[
\nabla f(1, 1) = 2\vec{i} - \vec{j},
\]

so one possible answer is \( 2\vec{i} - \vec{j} \).

(b) The line is

\[
2(x - 1) - (y - 1) = 0, \quad \text{or} \quad 2x - y = 1.
\]
(c) The graph is the level surface \( g(x, y, z) = 0 \) of the function \( g(x, y, z) = f(x, y) - z \). The gradient of \( g \) is normal to the level surface at each point. We have
\[
\nabla g(x, y, z) = \nabla f(x, y) - \vec{k}.
\]
The point on the surface where \( (x, y) = (1, 1) \) is \( (1, 1, f(1, 1)) = (1, 1, 1) \). A vector normal to the graph at \( (1, 1, 1) \) is
\[
\nabla g(1, 1, 1) = \nabla f(1, 1) - \vec{k} = 2\vec{i} - \vec{j} - \vec{k}.
\]

(d) The plane is
\[
2(x - 1) - (y - 1) - (z - 1) = 0, \quad \text{or} \quad 2x - y - z = 0.
\]

6. (a) \( \nabla f(1, 3) = f_x(1, 3)\vec{i} + f_y(1, 3)\vec{j} = 2\vec{i} + 4\vec{j} \)

(b) \( 2\vec{i} + 4\vec{j} \) (from (a); the gradient vector at a point is perpendicular to the contour line through that point)

(c) The graph is the level surface \( g(x, y, z) = 0 \) of the function \( g(x, y, z) = f(x, y) - z \). The gradient of \( g \) is normal to the level surface at each point. We have
\[
\nabla g(x, y, z) = \nabla f(x, y) - \vec{k}.
\]
The point on the surface where \( (x, y) = (1, 3) \) is \( (1, 3, f(1, 3)) = (1, 3, 1) \). A vector normal to the graph at \( (1, 3, 1) \) is
\[
\nabla g(1, 3, 1) = \nabla f(1, 3) - \vec{k} = 2\vec{i} + 4\vec{j} - \vec{k}.
\]

(d) \( \vec{u} = (\vec{i} + \vec{j})/\sqrt{2} \) is a unit vector in the direction of \( \vec{i} + \vec{j} \). The rate of change of \( f \) in this direction is \( f_{\vec{u}}(1, 3) = \nabla f(1, 3) \cdot \vec{u} = (2\vec{i} + 4\vec{j}) \cdot (\vec{i} + \vec{j})/\sqrt{2} = 6/\sqrt{2} = 3\sqrt{2} \).

(e) Near \( (1, 3) \), we have
\[
f(x, y) \approx f(1, 3) + f_x(1, 3)(x - 1) + f_y(1, 3)(y - 3) + \frac{f_{xx}(1, 3)}{2}(x - 1)^2 + f_{xy}(1, 3)(x - 1)(y - 3) + \frac{f_{yy}(1, 3)}{2}(y - 3)^2.
\]
So
\[
f(1.2, 3.3) \approx 1 + (2)(0.2) + (4)(0.3) + (2/2)(0.2)^2 + (-1)(0.2)(0.3) + (4/2)(0.3)^2 = 2.76.
\]

7. (a) In vector form, the equation of a line is \( \vec{r} = \vec{r}_0 + t\vec{v} \), where \( \vec{r}_0 \) is the position vector of a point in the line, and \( \vec{v} \) is a vector in the direction of the line. We already have \( \vec{r}_0 = \vec{i} + \vec{j} + \vec{k} \). Let \( f(x, y, z) = x^2 + y^2 + 2z^2 \). Since the gradient vector of \( f \) is perpendicular to the level surface, we can use it for \( \vec{v} \). That is, \( \vec{v} = \nabla f(1, 1, 1) = 2\vec{i} + 2\vec{j} + 4\vec{k} \). Thus the equation of the line is
\[
\vec{r} = \vec{i} + \vec{j} + \vec{k} + t(2\vec{i} + 2\vec{j} + 4\vec{k}),
\]
or
\[
x = 1 + 2t, \quad y = 1 + 2t, \quad z = 1 + 4t.
\]
(b) We can find the points by first finding the values of $t$ at which the line intersects the surface $x^2 + y^2 + 2z^2 = 4$. Plugging the parametric equations into the equation of the surface, we have

\[(1 + 2t)^2 + (1 + 2t)^2 + 2(1 + 4t)^2 = 4\]

\[40t^2 + 24t + 4 = 4\]

\[t(5t + 3) = 0\]

so $t = 0$ or $t = -3/5$. At $t = 0$, the parametric equations of the line give the point $(1, 1, 1)$, which is the point we already knew. At $t = -3/5$, the parametric equations of the line give $(-1/5, -1/5, -7/5)$. This is the other point that we want.

8. (a) First get the numbers:

\[f(1,2) = -1 + 4 + 1 - 2 = 2,\]

\[f_x(x, y) = 3(x - y)^2 + 2y + 2x, \quad f_x(1, 2) = 3 + 4 + 2 = 9,\]

\[f_y(x, y) = -3(x - y)^2 + 2x - 1, \quad f_y(1, 2) = -3 + 2 - 1 = -2.\]

Then

\[L(x, y) = f(1, 2) + f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2)\]

\[= 2 + 9(x - 1) - 2(y - 2).\]

(b) We need some more numbers:

\[f_{xx}(x, y) = 6(x - y) + 2, \quad f_{xx}(1, 2) = -6 + 2 = -4,\]

\[f_{xy}(x, y) = -6(x - y) + 2, \quad f_{xy}(1, 2) = 6 + 2 = 8,\]

\[f_{yy}(x, y) = 6(x - y), \quad f_{yy}(1, 2) = -6.\]

Then

\[Q(x, y) = L(x, y) + \frac{f_{xx}(1, 2)}{2}(x - 1)^2 + f_{xy}(1, 2)(x - 1)(y - 2) + \frac{f_{yy}(1, 2)}{2}(y - 2)^2\]

\[= 2 + 9(x - 1) - 2(y - 2) - 2(x - 1)^2 + 8(x - 1)(y - 2) - 3(y - 2)^2.\]

9. (a) This function is not differentiable on the circle $x^2 + y^2 = 1$. The graph has a “corner” at these points.

(b) This function is not differentiable at the origin. Consider the cross section $y = 0$:

\[f(x, 0) = (x^2)^{1/3} = \sqrt[3]{|x|}.\]

The graph has a cusp (i.e. a point) at $x = 0$.

(c) This function is the composition of polynomials and the exponential function, so it is differentiable everywhere.

(d) This function is not differentiable at points where the denominator is zero; that is, where $x^2 = y^2$. This gives the lines $y = x$ and $y = -x.$
10. (a) 

\[ \frac{dw}{dt} = Q_x(x, y, z) \frac{dx}{dt} + Q_y(x, y, z) \frac{dy}{dt} + Q_z(x, y, z) \frac{dz}{dt} \]

\[ = Q_x(f(t), g(t), h(t)) f'(t) + Q_y(f(t), g(t), h(t)) g'(t) + Q_z(f(t), g(t), h(t)) h'(t) \]

\[ = Q_x(\vec{r}(t)) f'(t) + Q_y(\vec{r}(t)) g'(t) + Q_z(\vec{r}(t)) h'(t) \]

(b) Since 

\[ \text{grad } Q(x, y, z) = Q_x(x, y, z) \vec{i} + Q_y(x, y, z) \vec{j} + Q_z(x, y, z) \vec{k}, \]

we have 

\[ \text{grad } Q(\vec{r}(t)) = Q_x(\vec{r}(t)) \vec{i} + Q_y(\vec{r}(t)) \vec{j} + Q_z(\vec{r}(t)) \vec{k}, \]

Also, 

\[ \frac{d\vec{r}}{dt} = f'(t) \vec{i} + g'(t) \vec{j} + h'(t) \vec{k} \]

so we have 

\[ \frac{dw}{dt} = Q_x(\vec{r}(t)) f'(t) + Q_y(\vec{r}(t)) g'(t) + Q_z(\vec{r}(t)) h'(t) \]

\[ = (\text{grad } Q(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}) \]