Math 113 – Calculus III **Exam 3 Practice Problems** **Fall 2005**

1. Suppose the motion of a particle is given by

$$x = 4\cos t, \qquad y = \sin t.$$

   (a) Describe the motion of the particle, and sketch the curve along which the particle travels.

   (b) Find the velocity and acceleration vectors of the particle.

   (c) Find the times t and the points on the curve where the speed of the particle is greatest.

   (d) Find the times t and the points on the curve where the magnitude of the acceleration is greatest.

2. Find the coordinates of the points where the line

$$x = t, \quad y = 1 + t, \quad z = 5t$$

intersects the surface

$$z = x^2 + y^2.$$

3. Let

$$g(x, y, z) = e^{-(x+y)^2} + z^2(x + y).$$

Suppose that a piece of fruit is sitting on a table in a room, and at each point (x, y, z) in the space within the room, $g(x, y, z)$ gives the strength of the odor of the fruit. Furthermore, suppose that a certain bug always flies in the direction in which the fruit odor increases fastest. Suppose also that the bug always flies with a *speed* of 2 feet/second.

What is the velocity vector of the bug when it is at the position $(2, -2, 1)$?

4. The path of a particle in space is given by the functions $x(t) = 2t$, $y(t) = \cos(t)$, and $z(t) = \sin(t)$. Suppose the temperature in this space is given by a function $H(x, y, z)$.

Find $\frac{dH}{dt}$, the rate of change of the temperature at the particle's position. (Since the actual function $H(x, y, z)$ is not given, your answer will be in terms of derivatives of H.)

5. Let

$$f(x, y) = x^3 - xy + \cos(\pi(x + y)).$$

(a) Find a vector normal to the level curve $f(x, y) = 1$ at the point where $x = 1$, $y = 1$.

(b) Find the equation of the line tangent to the level curve $f(x, y) = 1$ at the point where $x = 1$, $y = 1$.

(c) Find a vector normal to the graph $z = f(x, y)$ at the point $x = 1$, $y = 1$.

(d) Find the equation of the plane tangent to the graph $z = f(x, y)$ at the point $x = 1$, $y = 1$.

6. Suppose f is a differentiable function such that

$$f(1, 3) = 1, \quad f_x(1, 3) = 2, \quad f_y(1, 3) = 4,$$

$$f_{xx}(1, 3) = 2, \quad f_{xy}(1, 3) = -1, \quad \text{and} \quad f_{yy}(1, 3) = 4.$$

(a) Find $\text{grad} f(1, 3)$.

(b) Find a vector in the plane that is perpendicular to the contour line $f(x, y) = 1$ at the point $(1, 3)$.

(c) Find a vector that is perpendicular to the surface $z = f(x, y)$ (i.e. the graph of f) at the point $(1, 3, 1)$.

(d) At the point $(1, 3)$, what is the rate of change of f in the direction $\vec{i} + \vec{j}$?

(e) Use a quadratic approximation to estimate $f(1.2, 3.3)$.

7. We say that a line in 3-space is *normal to a surface* at a point of intersection if the line is normal to (i.e. perpendicular to) the tangent plane of the surface at that point.

Let S be the surface defined by

$$x^2 + y^2 + 2z^2 = 4.$$

(a) Find the parametric equations of the line that is normal to the surface S at the point $(1, 1, 1)$.

(b) The line found in (a) will intersect the surface S at two points. One of them is $(1, 1, 1)$, by construction. Find the other point of intersection.

8. Let

$$f(x, y) = (x - y)^3 + 2xy + x^2 - y.$$

(a) Find the function $L(x, y)$ that gives the linear approximation of f near the point $(1, 2)$.

(b) Find the function $Q(x, y)$ that gives the quadratic approximation of f near the point $(1, 2)$

9. For each of the following functions, determine the set of points where the function is *not* differentiable. *Briefly* explain how you know it is not differentiable; use a picture if it helps.

   (You do not have to *prove* that it is not differentiable; just identify the set of points based on your understanding of what differentiable means.)

   (a) $f(x,y) = \left|x^2 + y^2 - 1\right|$

   (b) $f(x,y) = (x^2 + y^2)^{1/4}$

   (c) $f(x,y) = e^{-x^2+y}$

   (d) $f(x,y) = \dfrac{x^3 - xy + 1}{x^2 - y^2}$

10. Suppose $w = Q(x,y,z)$, where Q is a differentiable function. Next suppose that $x = f(t)$, $y = g(t)$ and $z = h(t)$.

    (a) Use the chain rule to find an expression for $\frac{dw}{dt}$ in terms of Q, f, g, h and their derivatives (e.g. Q_x, f', etc.).

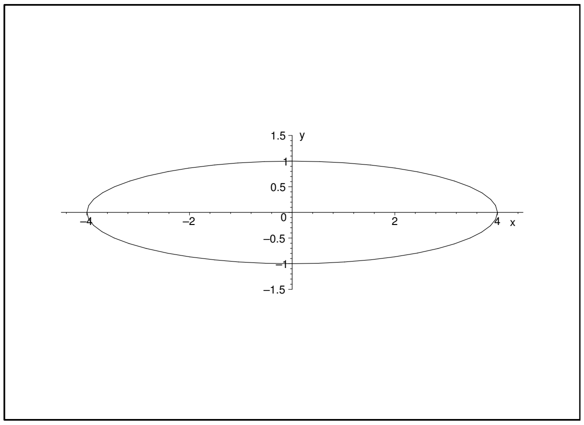
    (b) Show that the expression in (a) may be written as

    $$\frac{dw}{dt} = (\text{grad}\, Q(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt},$$

    where $\cdot$ is the dot product, $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ is the vector form of the parameterized curve $x = f(t)$, $y = g(t)$, and $z = h(t)$, and, if $\vec{r} = a\vec{i} + b\vec{j} + c\vec{k}$, $Q(\vec{r})$ means $Q(a,b,c)$.

SOLUTIONS

1. (a) Note that $(x/4)^2+y^2=\cos^2 t+\sin^2 t=1$, so the path of the particle is the *ellipse* $(x/4)^2+y^2=1$. The motion is counter-clockwise around the ellipse.



(b) The path is $\vec{r}(t)=(4\cos t)\vec{i}+(\sin t)\vec{j}$, so the velocity is

$$\vec{v}(t)=\vec{r}\,'(t)=(-4\sin t)\vec{i}+(\cos t)\vec{j}$$

and the acceleration is

$$\vec{a}(t)=\vec{r}\,''(t)=(-4\cos t)\vec{i}+(-\sin t)\vec{j}.$$

(c) The speed is

$$v=\|\vec{v}(t)\|=\sqrt{(-4\sin t)^2+(\cos t)^2}=\sqrt{16\sin^2 t+\cos^2 t}.$$

This has a maximum or minimum when $dv/dt=0$. Now

$$\frac{dv}{dt}=\frac{15\sin t\cos t}{\sqrt{16\sin^2 t+\cos^2 t}},$$

so $dv/dt=0$ when $\sin t=0$ or $\cos t=0$. This means $t=n\pi$ or $t=\pi/2+n\pi$, where n is any integer. When $t=n\pi$, $v=1$, and when $t=\pi/2+n\pi$, $v=4$. Thus the maximum speed is $v=4$. It occurs when $t=\pi/2+n\pi$. When n is even, the point in the plane where this occurs is $(0,1)$, and when n is odd, the point is $(0,-1)$.

(d) The magnitude of the acceleration is

$$a=\|\vec{a}(t)\|=\sqrt{(-4\cos t)^2+(-\sin t)^2}=\sqrt{16\cos^2 t+\sin^2 t}.$$

This has a maximum or minimum when $da/dt=0$. Now

$$\frac{da}{dt}=\frac{-15\sin t\cos t}{\sqrt{16\cos^2 t+\sin^2 t}},$$

so $da/dt = 0$ when $\sin t = 0$ or $\cos t = 0$. This means $t = n\pi$ or $t = \pi/2 + n\pi$, where n is any integer. When $t = n\pi$, $a = 4$, and when $t = \pi/2 + n\pi$, $a = 1$. Thus the maximum magnitude of the acceleration is $a = 4$. It occurs when $t = n\pi$. When n is even, the point in the plane where this occurs is $(0, 4)$, and when n is odd, the point is $(0, -4)$.

2. We have $x = t$, $y = 1 + t$ and $z = 5t$, and we want $z = x^2 + y^2$, so we must solve

$$5t = t^2 + (1+t)^2 \implies 2t^2 - 3t + 1 = 0 \implies t = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4}.$$

So the line intersects the surface when $t = 1/2$ and when $t = 1$. When $t = 1/2$, the point of intersection is $(1/2, 3/2, 5/2)$, and when $t = 1$, the point of intersection is $(1, 2, 5)$.

3. Since the bug flies in the direction in which the fruit odor increases fastest, it flies in the direction of $\operatorname{grad} g$. The gradient of g is

$$\operatorname{grad} g(x,y,z) = \left(-2(x+y)e^{-(x+y)^2} + z^2\right)\vec{i} + \left(-2(x+y)e^{-(x+y)^2} + z^2\right)\vec{j} + \left(2z(x+y)\right)\vec{k},$$

and

$$\operatorname{grad} g(2,-2,1) = \vec{i} + \vec{j}.$$

The bug always has a speed of 2, so the velocity vector must have a magnitude of 2. A vector with magnitude 2 and in the same direction as the gradient is

$$2\frac{\operatorname{grad} g(2,-2,1)}{\|\operatorname{grad} g(2,-2,1)\|} = \frac{2}{\sqrt{2}}(\vec{i} + \vec{j}).$$

4. By the chain rule,

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} + \frac{\partial H}{\partial z}\frac{dz}{dt} = 2\frac{\partial H}{\partial x} - \sin t\frac{\partial H}{\partial y} + \cos t\frac{\partial H}{\partial z}$$

5. (a) The gradient of f is normal to the level curve at each point. We find

$$\operatorname{grad} f(x,y) = (3x^2 - y - \pi\sin(\pi(x+y)))\vec{i} + (-x - \pi\sin(\pi(x+y)))\vec{j},$$

and

$$\operatorname{grad} f(1,1) = 2\vec{i} - \vec{j},$$

so one possible answer is $2\vec{i} - \vec{j}$.

(b) The line is

$$2(x-1) - (y-1) = 0, \quad \text{or} \quad 2x - y = 1.$$

(c) The graph is the level surface $g(x, y, z) = 0$ of the function $g(x, y, z) = f(x, y) - z$. The gradient of g is normal to the level surface at each point. We have $\text{grad}\, g(x, y, z) = \text{grad}\, f(x, y) - \vec{k}$. The point on the surface where $(x, y) = (1, 1)$ is $(1, 1, f(1, 1)) = (1, 1, 1)$. A vector normal to the graph at $(1, 1, 1)$ is

$$\text{grad}\, g(1, 1, 1) = \text{grad}\, f(1, 1) - \vec{k} = 2\vec{i} - \vec{j} - \vec{k}.$$

(d) The plane is $\quad 2(x - 1) - (y - 1) - (z - 1) = 0, \quad$ or $\quad 2x - y - z = 0.$

6. (a) $\text{grad} f(1, 3) = f_x(1, 3)\vec{i} + f_y(1, 3)\vec{j} = 2\vec{i} + 4\vec{j}$

(b) $2\vec{i} + 4\vec{i}$ (from (a); the gradient vector at a point is perpendicular to the contour line through that point)

(c) The graph is the level surface $g(x, y, z) = 0$ of the function $g(x, y, z) = f(x, y) - z$. The gradient of g is normal to the level surface at each point. We have $\text{grad}\, g(x, y, z) = \text{grad}\, f(x, y) - \vec{k}$. The point on the surface where $(x, y) = (1, 3)$ is $(1, 3, f(1, 3)) = (1, 3, 1)$. A vector normal to the graph at $(1, 3, 1)$ is

$$\text{grad}\, g(1, 3, 1) = \text{grad}\, f(1, 3) - \vec{k} = 2\vec{i} + 4\vec{j} - \vec{k}.$$

(d) $\vec{u} = (\vec{i} + \vec{j})/\sqrt{2}$ is a unit vector in the direction of $\vec{i} + \vec{j}$. The rate of change of f in this direction is $f_{\vec{u}}(1, 3) = \text{grad} f(1, 3) \cdot \vec{u} = (2\vec{i} + 4\vec{j}) \cdot (\vec{i} + \vec{j})/\sqrt{2} = 6/\sqrt{2} = 3\sqrt{2}$.

(e) Near $(1, 3)$, we have

$$f(x, y) \approx f(1, 3) + f_x(1, 3)(x - 1) + f_y(1, 3)(y - 3) + \frac{f_{xx}(1, 3)}{2}(x - 1)^2 + f_{xy}(1, 3)(x - 1)(y - 3) + \frac{f_{yy}(1, 3)}{2}(y - 3)^2.$$

So

$$\begin{aligned} f(1.2, 3.3) &\approx 1 + (2)(0.2) + (4)(0.3) + (2/2)(0.2)^2 + (-1)(0.2)(0.3) + (4/2)(0.3)^2 \\ &= 2.76. \end{aligned}$$

7. (a) In vector form, the equation of a line is $\vec{r} = \vec{r}_0 + t\vec{v}$, where $\vec{r}_0$ is the position vector of a point in the line, and $\vec{v}$ is a vector in the direction of the line. We already have $\vec{r}_0 = \vec{i} + \vec{j} + \vec{k}$. Let $f(x, y, z) = x^2 + y^2 + 2z^2$. Since the gradient vector of f is perpendicular to the level surface, we can use it for $\vec{v}$. That is, $\vec{v} = \text{grad}\, f(1, 1, 1) = 2\vec{i} + 2\vec{j} + 4\vec{k}$. Thus the equation of the line is

$$\vec{r} = \vec{i} + \vec{j} + \vec{k} + t(2\vec{i} + 2\vec{j} + 4\vec{k}),$$

or

$$x = 1 + 2t, \quad y = 1 + 2t, \quad z = 1 + 4t.$$

(b) We can find the points by first finding the values of t at which the line intersects the surface $x^2+y^2+2z^2=4$. Plugging the parametric equations into the equation of the surface, we have

$$\begin{aligned}(1+2t)^2+(1+2t)^2+2(1+4t)^2&=4\\ 40t^2+24t+4&=4\\ t(5t+3)&=0\end{aligned}$$

so $t=0$ or $t=-3/5$. At $t=0$, the parametric equations of the line give the point $(1,1,1)$, which is the point we already knew. At $t=-3/5$, the parametric equations of the line give $(-1/5,-1/5,-7/5)$. This is the other point that we want.

8. (a) First get the numbers:
$f(1,2)=-1+4+1-2=2$,
$f_x(x,y)=3(x-y)^2+2y+2x$, $\quad f_x(1,2)=3+4+2=9$,
$f_y(x,y)=-3(x-y)^2+2x-1$, $\quad f_y(1,2)=-3+2-1=-2$.
Then

$$\begin{aligned}L(x,y)&=f(1,2)+f_x(1,2)(x-1)+f_y(1,2)(y-2)\\ &=2+9(x-1)-2(y-2).\end{aligned}$$

(b) We need some more numbers:
$f_{xx}(x,y)=6(x-y)+2$, $\quad f_{xx}(1,2)=-6+2=-4$,
$f_{xy}(x,y)=-6(x-y)+2$, $\quad f_{xy}(1,2)=6+2=8$,
$f_{yy}(x,y)=6(x-y)$, $\quad f_{yy}(1,2)=-6$.
Then

$$\begin{aligned}Q(x,y)&=L(x,y)+\frac{f_{xx}(1,2)}{2}(x-1)^2+f_{xy}(1,2)(x-1)(y-2)+\frac{f_{yy}(1,2)}{2}(y-2)^2\\ &=2+9(x-1)-2(y-2)-2(x-1)^2+8(x-1)(y-2)-3(y-2)^2.\end{aligned}$$

9. (a) This function is not differentiable on the circle $x^2+y^2=1$. The graph has a "corner" at these points.

(b) This function is not differentiable at the origin. Consider the cross section $y=0$: $f(x,0)=(x^2)^{1/4}=\sqrt{|x|}$. The graph has a cusp (i.e. a point) at $x=0$.

(c) This function is the composition of polynomials and the exponential function, so it is differentiable everywhere.

(d) This function is not differentiable at points where the denominator is zero; that is, where $x^2=y^2$. This gives the lines $y=x$ and $y=-x$.

10. (a)

$$\begin{aligned}\frac{dw}{dt} &= Q_x(x,y,z)\frac{dx}{dt} + Q_y(x,y,z)\frac{dy}{dt} + Q_z(x,y,z)\frac{dz}{dt} \\ &= Q_x(f(t),g(t),h(t))f'(t) + Q_y(f(t),g(t),h(t))g'(t) + Q_z(f(t),g(t),h(t))h'(t) \\ &= Q_x(\vec{r}(t))f'(t) + Q_y(\vec{r}(t))g'(t) + Q_z(\vec{r}(t))h'(t)\end{aligned}$$

(b) Since

$$\operatorname{grad} Q(x,y,z) = Q_x(x,y,z)\vec{i} + Q_y(x,y,z)\vec{j} + Q_z(x,y,z)\vec{k},$$

we have

$$\operatorname{grad} Q(\vec{r}(t)) = Q_x(\vec{r}(t))\vec{i} + Q_y(\vec{r}(t))\vec{j} + Q_z(\vec{r}(t))\vec{k},$$

Also,

$$\frac{d\vec{r}}{dt} = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$$

so we have

$$\begin{aligned}\frac{dw}{dt} &= Q_x(\vec{r}(t))f'(t) + Q_y(\vec{r}(t))g'(t) + Q_z(\vec{r}(t))h'(t) \\ &= (\operatorname{grad} Q(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\end{aligned}$$