Math 113 – Calculus III **Exam 1 Practice Problems** **Fall 2005**

Name: ____

1. Match each function to one of the given graphs, and to one of the given contour diagrams. The plots are given in a separate handout.

$f(x,y)$	Graph	Contour Diagram
$x^2 + \left(\frac{3y}{2}\right)^2$		
$x^2 - \frac{1}{2}y^2$		
$x + \frac{y}{2}$		
$-x + \frac{y}{2}$		
$\sin(\pi x)$		
xy^2		
$\sqrt{x^2 + \left(\frac{3y}{2}\right)^2}$		
x^2		

2. (a) Consider the function

$$f(x, y) = x^2 - y - xe^y - 1.$$

Find a function $g(x, y, z)$ such that the graph of f is the level surface $g(x, y, z) = 5$.

$g(x, y, z) =$ ____________________

(b) For each of the following functions, determine if the level surface $g(x, y, z) = 0$ can be expressed as the graph of a function $f(x, y)$. If it is not possible, explain why not. If is it possible, find the function $f(x, y)$.

i. $g(x, y, z) = x^2 + x + y^4 + z(z - 1)$

ii. $g(x, y, z) = \sin(z - y + 2x)$

iii. $g(x, y, z) = 1 - e^{x^2 - y + z}$

3. Describe the level surfaces of the function $g(x, y, z) = x^2 + 4y^2 + z$.

4. Determine the points (if there are any) where the following functions are *not* continuous. Justify your answers.

(a) $f(x, y) = \dfrac{\sin(x + y)}{x - y}$

(b) $g(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

(c) $h(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0 \\ 0 & (x, y) = (0, 0) \end{cases}$

SOLUTIONS

1.

$f(x,y)$	Graph	Contour Diagram
$x^2 + \left(\frac{3y}{2}\right)^2$	**C**	**8**
$x^2 - \frac{1}{2}y^2$	**A**	**4**
$x + \frac{y}{2}$	**B**	**7**
$-x + \frac{y}{2}$	**F**	**5**
$\sin(\pi x)$	**H**	**1**
xy^2	**E**	**6**
$\sqrt{x^2 + \left(\frac{3y}{2}\right)^2}$	**G**	**2**
x^2	**D**	**3**

2. (a) **The graph of f is given by**

$$z = x^2 - y - xe^y - 1,$$

which we may rewrite as $x^2 - y - xe^y - z - 1 = 0$, or, by adding 5 to both sides,

$$x^2 - y - xe^y - z + 4 = 5.$$

So we can choose

$$g(x, y, z) = x^2 - y - xe^y - z + 4.$$

(b) i. $g(x, y, z) = x^2 + x + y^4 + z(z - 1)$
NO. If we try to solve $g(x, y, z) = 0$ for z, we first find

$$z^2 - z + x^2 + x + y^4 = 0,$$

and to solve this for z, we have to use the quadratic formula:

$$z = \frac{1}{2}\left(1 \pm \sqrt{1 - 4(x^2 + x + y^4)}\right),$$

which means there will be two z values for each point (x, y) where the expression inside the square root is not negative. So we can not express the surface as the graph of a single function $f(x, y)$.

ii. $g(x, y, z) = \sin(z - y + 2x)$
NO. The level surface $g(x, y, z) = 0$ is $\sin(z - y + 2x) = 0$, which implies $z - y + 2x = n\pi$, where n is some integer. That is, the level set $g(x, y, z) = 0$ consists of infinitely many planes of the form

$$z = -2x + y + n\pi.$$

Since there is not a single function of x and y whose graph is this level set, the surface can not be expressed as the graph of a function $f(x, y)$.

iii. $g(x, y, z) = 1 - e^{x^2 - y + z}$
YES. Solve for z: $g(x, y, z) = 0 \implies 1 - e^{x^2-y+z} = 0 \implies e^{x^2-y+z} = 1$ $\implies x^2 - y + z = \ln(1) = 0 \implies z = y - x^2$. **So the surface $g(x, y, z) = 0$ is the graph of $f(x, y) = y - x^2$.**

3. **The level surfaces are given by $x^2 + 4y^2 + z = c$ for some constant c. We can rewrite this as $z = -x^2 - 4y^2 + c$. These surfaces are elliptical paraboloids that open downwards. (That is, they are "dome-shaped", with parabolas for cross-sections, and ellipses for contour lines.) Each level surface has the same shape. The constant c gives the z intercept of each paraboloid.**

4. (a) $f(x, y) = \dfrac{\sin(x + y)}{x - y}$

The numerator and denominator are both continuous, so the only points where the function would not be continuous are where the denominator is 0. So this function is not continuous along the line $y = x$. (In fact, the function is not defined there.)

(b) $g(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

The numerator and denominator are both continuous, and the denominator is never 0, so there are no points where this function is not continuous.

(c) $h(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0 \\ 0 & (x, y) = (0, 0) \end{cases}$

For $(x, y) \neq (0, 0)$, the function is a quotient of polynomials, and the denominator is not 0 if $(x, y) \neq (0, 0)$, so the function is continuous for all $(x, y) \neq (0, 0)$. To determine if the function is continuous at $(0, 0)$, we must first determine if $\lim_{(x,y)\to(0,0)} h(x, y)$ exists. Let's check a few paths towards $(0, 0)$. Along the x axis, we have $y = 0$, and

$$\lim_{x\to 0} h(x, 0) = \lim_{x\to 0} \frac{0}{x^2} = 0.$$

Along the y axis, we have $x = 0$, and we find the same limit, 0. Let's try along the line $y = x$. Then

$$\lim_{x\to 0} h(x, x) = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

We see that the function approaches the value 0 along the x axis (and along the y axis), but approaches the value $1/2$ along the line $y = x$. Since the function approaches different values along two different paths to $(0, 0)$, the limit $\lim_{(x,y)\to(0,0)} h(x, y)$ does not exist. Therefore the function is not continuous at $(0, 0)$.