Math 113 – Calculus III SOLUTIONS: Exam 2 Practice Problems Spring 2003

1. Suppose $\vec{u}$ is a unit vector, and $\vec{v}$ and $\vec{w}$ are two more vectors that are not necessarily unit vectors. Simplify the following expression as much as possible:

$$((\vec{v}\cdot\vec{u})\vec{u})\cdot(\vec{v}\times\vec{w}) - (\vec{w}\times\vec{v})\cdot(\vec{v}-(\vec{u}\cdot\vec{v})\vec{u}).$$

$$\begin{aligned}
&((\vec{v}\cdot\vec{u})\vec{u})\cdot(\vec{v}\times\vec{w}) - (\vec{w}\times\vec{v})\cdot(\vec{v}-(\vec{u}\cdot\vec{v})\vec{u}) \\
&\qquad = ((\vec{v}\cdot\vec{u})\vec{u})\cdot(\vec{v}\times\vec{w}) + (\vec{v}\times\vec{w})\cdot(\vec{v}-(\vec{v}\cdot\vec{u})\vec{u}) \\
&\qquad = [(\vec{v}\cdot\vec{u})\vec{u}+\vec{v}-(\vec{v}\cdot\vec{u})\vec{u}]\cdot(\vec{v}\times\vec{w}) \\
&\qquad = \vec{v}\cdot(\vec{v}\times\vec{w}) \\
&\qquad = 0.
\end{aligned}$$

2. Let $P=(1,1,1)$, $Q=(1,-3,0)$ and $R=(2,2,2)$.

   (a) Find the equation of the plane that contains the points P, Q, and R.

   We have a point in the plane (in fact, we have three). All we need is a normal vector. This is given by

$$\begin{aligned}
\vec{n} &= \overrightarrow{PQ}\times\overrightarrow{PR} \\
&= \left((1-1)\vec{i}+(-3-1)\vec{j}+(0-1)\vec{k}\right)\times\left((2-1)\vec{i}+(2-1)\vec{j}+(2-1)\vec{k}\right) \\
&= \left(-4\vec{j}-\vec{k}\right)\times\left(\vec{i}+\vec{j}+\vec{k}\right) \\
&= -3\vec{i}-\vec{j}+4\vec{k}.
\end{aligned}$$

   By using $\vec{n}$ and the point P, we find the equation of the plane to be

$$-3(x-1)-(y-1)+4(z-1)=0 \quad \text{or} \quad -3x-y+4z=0.$$

   (b) Find the area of the triangle formed by the three points.

   $\frac{\|\vec{n}\|}{2} = \frac{\sqrt{9+1+16}}{2} = \frac{\sqrt{26}}{2}$

   (c) Find the distance from the plane found in (a) to the point $(3,4,5)$.

   Let R be the point $(3,4,5)$.
   Let $\vec{u} = \frac{\vec{n}}{\|\vec{n}\|} = \frac{1}{\sqrt{26}}\left(-3\vec{i}-\vec{j}+4\vec{k}\right)$.
   Then $d = \overrightarrow{PR}\cdot\vec{u} = \frac{1}{\sqrt{26}}\left(2\vec{i}+3\vec{j}+4\vec{k}\right)\cdot\left(-3\vec{i}-\vec{j}+4\vec{k}\right) = \frac{7}{\sqrt{26}}$.

3. We say two planes are perpendicular if their normal vectors are perpendicular. Given the following two planes (which are *not* perpendicular):

$$x+2y+4z=1, \qquad -x+y-2z=5,$$

find the equation of a plane that is perpendicular to *both* of these planes, and that contains the point $(3, 2, 1)$.

Normal vectors for the two given planes are $\vec{n}_1 = \vec{i} + 2\vec{j} + 4\vec{k}$ and $\vec{n}_2 = -\vec{i} + \vec{j} - 2\vec{k}$, respectively.
Then $\vec{n}_3 = \vec{n}_1 \times \vec{n}_2 = -8\vec{i} - 2\vec{j} + 3\vec{k}$ is perpendicular to both $\vec{n}_1$ and $\vec{n}_2$, and therefore a plane with normal vector $\vec{n}_3$ will be perpendicular to the given planes. We are told that $(3, 2, 1)$ is a point in the plane, so the equation of the plane is

$$-8(x-3) - 2(y-2) + 3(z-1) = 0.$$

4. Let

$$g(x, y, z) = e^{-(x+y)^2} + z^2(x+y).$$

(a) What is the instantaneous rate of change of g at the point $(2, -2, 1)$ in the direction of the origin?

We want the directional derivative of g at $(2, -2, 1)$ in the direction of the origin. A vector in this direction is $-2\vec{i} + 2\vec{j} - \vec{k}$, and a unit vector in this direction is $\vec{u} = \frac{1}{\sqrt{9}}(-2\vec{i} + 2\vec{j} - \vec{k}) = \left(-\frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{1}{3}\vec{k}\right)$. The gradient of g is

$$\operatorname{grad} g(x, y, z) = \left(-2(x+y)e^{-(x+y)^2} + z^2\right)\vec{i} + \left(-2(x+y)e^{-(x+y)^2} + z^2\right)\vec{j} + (2z(x+y))\,\vec{k},$$

and in particular

$$\operatorname{grad} g(2, -2, 1) = \vec{i} + \vec{j}.$$

Then the instantaneous rate of change of g in the direction $\vec{u}$ at the point $(2, -2, 1)$ is

$$g_{\vec{u}}(2, -2, 1) = \operatorname{grad} g(2, -2, 1) \cdot \vec{u} = \left(\vec{i} + \vec{j}\right) \cdot \left(-\frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{1}{3}\vec{k}\right) = 0.$$

(b) Suppose that a piece of fruit is sitting on a table in a room, and at each point (x, y, z) in the space within the room, $g(x, y, z)$ gives the strength of the odor of the fruit. Furthermore, suppose that a certain bug always flies in the direction in which the fruit odor increases fastest. Suppose also that the bug always flies with a *speed* of 2 feet/second.

What is the velocity vector of the bug when it is at the position $(2, -2, 1)$?

Since the bug flies in the direction in which the fruit odor increases fastest, it flies in the direction of $\operatorname{grad} g$. It always has a speed of 2, so the velocity vector at $(2, -2, 1)$ is

$$2\frac{\operatorname{grad} g(2, -2, 1)}{\|\operatorname{grad} g(2, -2, 1)\|} = \frac{2}{\sqrt{2}}(\vec{i} + \vec{j}).$$

5. The path of a particle in space is given by the functions $x(t) = 2t$, $y(t) = \cos(t)$, and $z(t) = \sin(t)$. Suppose the temperature in this space is given by a function $H(x, y, z)$.

   (a) Find $\frac{dH}{dt}$, the rate of change of the temperature at the particle's position. (Since the actual function $H(x, y, z)$ is not given, your answer will be in terms of derivatives of H.)

   $\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} + \frac{\partial H}{\partial z}\frac{dz}{dt} = 2\frac{\partial H}{\partial x} - \sin t\frac{\partial H}{\partial y} + \cos t\frac{\partial H}{\partial z}$

   (b) Suppose we know that at all points, $\frac{\partial H}{\partial x} > 0$, $\frac{\partial H}{\partial y} < 0$ and $\frac{\partial H}{\partial z} > 0$. At $t = 0$, is $\frac{dH}{dt}$ positive, zero, or negative?

   At $t = 0$, $\frac{dH}{dt} = 2\frac{\partial H}{\partial x} + \frac{\partial H}{\partial z} > 0$.

6. Let

$$f(x, y) = x^3 - xy + \cos(\pi(x + y)).$$

   (a) Find a vector normal to the level curve $f(x, y) = 1$ at the point where $x = 1$, $y = 1$.

   The gradient of f is normal to the level curve at each point. We find $\operatorname{grad} f(x, y) = (3x^2 - y - \pi\sin(\pi(x + y)))\vec{i} + (-x - \pi\sin(\pi(x + y)))\vec{j}$, and $\operatorname{grad} f(1, 1) = 2\vec{i} - \vec{j}$.

   (b) Find the equation of the line tangent to the level curve $f(x, y) = 1$ at the point where $x = 1$, $y = 1$.

   The line is

$$2(x - 1) - (y - 1) = 0, \quad \text{or} \quad 2x - y = 1.$$

   (c) Find a vector normal to the graph $z = f(x, y)$ at the point $x = 1$, $y = 1$.

   The graph is the level surface $g(x, y, z) = 0$ of the function $g(x, y, z) = f(x, y) - z$. The gradient of g is normal to the level surface at each point. We have $\operatorname{grad} g(x, y, z) = \operatorname{grad} f(x, y) - \vec{k}$. Now $f(1, 1) = 1$, so a vector normal to the graph at $(1, 1, 1)$ is

$$\operatorname{grad} g(1, 1, 1) = \operatorname{grad} f(1, 1) - \vec{k} = 2\vec{i} - \vec{j} - \vec{k}.$$

   (d) Find the equation of the plane tangent to the graph $z = f(x, y)$ at the point $x = 1$, $y = 1$.

   The plane is $2(x - 1) - (y - 1) - (z - 1) = 0$, or $2x - y - z = 0$.

7. Let

$$f(x, y) = (x - y)^3 + 2xy + x^2 - y.$$

   (a) Find the linear approximation $L(x, y)$ near the point $(1, 2)$.

First get the numbers: $f(1,2) = -1+4+1-2 = 2$,
$f_x(x,y) = 3(x-y)^2 + 2y + 2x$, $f_x(1,2) = 3+4+2 = 9$,
$f_y(x,y) = -3(x-y)^2 + 2x - 1$, $f_y(1,2) = -3+2-1 = -2$.
Then $L(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = 2 + 9(x-1) - 2(y-2)$.

(b) Find the quadratic approximation $Q(x,y)$ near the point $(1,2)$.

We need some more numbers:
$f_{xx}(x,y) = 6(x-y) + 2$, $f_{xx}(1,2) = -6+2 = -4$,
$f_{xy}(x,y) = -6(x-y) + 2$, $f_{xy}(1,2) = 6+2 = 8$,
$f_{yy}(x,y) = 6(x-y)$, $f_{xy}(1,2) = -6$.
Then

$$\begin{aligned} Q(x,y) &= L(x,y) + \frac{f_{xx}(1,2)}{2}(x-1)^2 + f_{xy}(1,2)(x-1)(y-2) + \frac{f_{yy}(1,2)}{2}(y-2)^2 \\ &= 2 + 9(x-1) - 2(y-2) - 2(x-1)^2 + 8(x-1)(y-2) - 3(y-2)^2. \end{aligned}$$

8. For each of the following functions, determine the set of points where the function is *not* differentiable. *Briefly* explain how you know it is not differentiable; use a picture if it helps.

(You do not have to *prove* that it is not differentiable; just identify the set of points based on your understanding of what differentiable means.)

(a) $f(x,y) = \left|x^2 + y^2 - 1\right|$

This function is not differentiable on the circle $x^2 + y^2 = 1$. The graph has a "corner" at these points.

(b) $f(x,y) = (x^2 + y^2)^{1/4}$

This function is not differentiable at the origin. Consider the cross section $y = 0$: $f(x,0) = (x^2)^{1/4} = \sqrt{|x|}$. The graph has a cusp (i.e. a point) at $x = 0$.

(c) $f(x,y) = e^{-x^2+y}$

This function is the composition of polynomials and the exponential function, so it is differentiable everywhere.

(d) $f(x,y) = \dfrac{x^3 - xy + 1}{x^2 - y^2}$

This function is not differentiable at points where the denominator is zero; that is, where $x^2 = y^2$. This gives the lines $y = x$ and $y = -x$.

9. Let $H(x,y) = x^2 - y^2 + xy$, and suppose that x and y are both functions that depend on t. Express $\dfrac{dH}{dt}$ in terms of x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} = (2x+y)\frac{dx}{dt} + (-2y+x)\frac{dy}{dt}$

10. Suppose f is a differentiable function such that

$$f(1,3)=1, \quad f_x(1,3)=2, \quad f_y(1,3)=4,$$

$$f_{xx}(1,3)=2, \quad f_{xy}(1,3)=-1, \quad \text{and} \quad f_{yy}(1,3)=4.$$

(a) Find $\operatorname{grad} f(1,3)$.

$\boxed{\operatorname{grad} f(1,3) = f_x(1,3)\vec{i} + f_y(1,3)\vec{j} = 2\vec{i} + 4\vec{j}}$

(b) Find a vector in the plane that is perpendicular to the contour line $f(x,y)=1$ at the point $(1,3)$.

$2\vec{i} + 4\vec{i}$ (from (a); the gradient vector at a point is perpendicular to the contour line through that point)

(c) Find a vector that is perpendicular to the surface $z = f(x,y)$ (i.e. the graph of f) at the point $(1,3,1)$.

The graph is the level surface $g(x,y,z)=0$ of the function $g(x,y,z) = f(x,y) - z$. The gradient of g is normal to the level surface at each point. We have $\operatorname{grad} g(x,y,z) = \operatorname{grad} f(x,y) - \vec{k}$. Now $f(1,3)=1$, so a vector normal to the graph at $(1,3,1)$ is

$$\operatorname{grad} g(1,3,1) = \operatorname{grad} f(1,3) - \vec{k} = 2\vec{i} + 4\vec{j} - \vec{k}.$$

(d) At the point $(1,3)$, what is the rate of change of f in the direction $\vec{i}+\vec{j}$?

$\vec{u} = (\vec{i}+\vec{j})/\sqrt{2}$ is a unit vector in the direction of $\vec{i}+\vec{j}$. The rate of change of f in this direction is $f_{\vec{u}}(1,3) = \operatorname{grad} f(1,3)\cdot\vec{u} = (2\vec{i}+4\vec{j})\cdot(\vec{i}+\vec{j})/\sqrt{2} = 6/\sqrt{2} = 3\sqrt{2}$.

(e) Use a quadratic approximation to estimate $f(1.2, 3.3)$.

Near $(1,3)$, we have

$$f(x,y) \approx f(1,3) + f_x(1,3)(x-1) + f_y(1,3)(y-3) + \frac{f_{xx}(1,3)}{2}(x-1)^2 + f_{xy}(1,3)(x-1)(y-3) + \frac{f_{yy}(1,3)}{2}(y-3)^2.$$

So $f(1.2, 3.3) \approx 1 + (2)(0.2) + (4)(0.3) + (2/2)(0.2)^2 + (-1)(0.2)(0.3) + (4/2)(0.3)^2 = 2.76$.