1. Let

$$g(x, y, z) = \sqrt{2x^2 + y^2 + 4z^2}.$$

(a) Describe the shapes of the level surfaces of g.

(b) In three different graphs, sketch the three cross sections to the level surface $g(x, y, z) = 1$ for which

   i. $x = 0$,

  ii. $y = 0$,

 iii. $z = 0$.

In each cross section, label the axes and any intercepts.

(c) Find the equation of the plane tangent to the surface $g(x, y, z) = 1$ at the point $\left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{4} \right)$.

2. Suppose f is a differentiable function such that

$$f(1, 3) = 1, \quad f_x(1, 3) = 2, \quad f_y(1, 3) = 4,$$

$$f_{xx}(1, 3) = 2, \quad f_{xy}(1, 3) = -1, \quad \text{and} \quad f_{yy}(1, 3) = 4.$$

(a) Find grad $f(1, 3)$.

(b) Find a vector in the plane that is perpendicular to the contour line $f(x, y) = 1$ at the point $(1, 3)$.

(c) Find a vector that is perpendicular to the surface $z = f(x, y)$ (i.e. the graph of f) at the point $(1, 3, 1)$.

(d) At the point $(1, 3)$, what is the rate of change of f in the direction $\vec{i} + \vec{j}$?

(e) Use a quadratic approximation to estimate $f(1.2, 3.3)$.

3. Let
$$f(x, y) = x^2 - 4x + y^2 - 4y + 16.$$

(a) Find and classify the critical points of f.

(b) Find the maximum and minimum values of f subject to the constraint
$$x^2 + y^2 = 18$$

(c) Find the maximum and minimum values of f subject to the constraint
$$x^2 + y^2 \leq 18$$

(d) Approximate the maximum value of f subject to the constraint
$$x^2 + y^2 = 18.3$$

(Explain your answer in terms of Lagrange multipliers.)

4. Suppose the integral of some function f over a region R in the plane is given in polar coordinates as
$$\int_0^3 \int_0^{\frac{\pi}{2}} r^2 \, d\theta dr.$$

(a) Sketch the region of integration R in the xy plane.

(b) Convert this integral to Cartesian coordinates.

(c) Evaluate the integral. (You may use either polar or Cartesian coordinates.)

5. Let W be the solid region where $x \geq 0$, $y \geq 0$, $z \geq 0$, $z \leq x + y$, and $x^2 + y^2 \leq 4$. (In other words, W is bounded by the yz plane, the xz plane, the xy plane, and the surfaces $z = x + y$ and $x^2 + y^2 = 4$.)

Let $f(x, y, z) = 1 + x + 2z$ be the density of the material in this region.

Express the total mass of the material in W as a triple integral in

(a) rectangular coordinates,

(b) cylindrical coordinates.

Your expressions should be complete enough that, in principle, they could be evaluated, but not evaluate the integrals!

6. We say that a line in 3-space is *normal to a surface* at a point of intersection if the line is normal to (i.e. perpendicular to) the tangent plane of the surface at that point.

   Let S be the surface defined by

   $$x^2 + y^2 + 2z^2 = 4.$$

   (a) Find the parametric equations of the line that is normal to the surface S at the point $(1, 1, 1)$.

   (b) The line found in (a) will intersect the surface S at two points. One of them is $(1, 1, 1)$, by construction. Find the other point of intersection.

7. Suppose $w = Q(x, y, z)$, where Q is a differentiable function. Next suppose that $x = f(t)$, $y = g(t)$ and $z = h(t)$.

   (a) Use the chain rule to find an expression for $\frac{dw}{dt}$ in terms of Q, f, g, h and their derivatives (e.g. Q_x, f', etc.).

   (b) Show that the expression in (a) may be written as

   $$\frac{dw}{dt} = (\operatorname{grad} Q) \cdot \frac{d\vec{r}}{dt},$$

   where $\cdot$ is the dot product, and $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ is the vector form of the parameterized curve $x = f(t)$, $y = g(t)$, and $z = h(t)$.

Brief Solutions

1. (a) The level set $g = 0$ is the point $(0,0,0)$. The level set $g = c$, where $c > 0$, is an ellipsoid.

   (b) Descriptions instead of plots: (i) an *ellipse* in the yz plane with y-intercepts $y = \pm 1$ and z-intercepts $z = \pm 1/2$; (ii) an *ellipse* in the xz plane with x-intercepts $x = \pm 1/\sqrt{2}$ and z-intercepts $z = \pm 1/2$; (iii) an *ellipse* in the xy plane with x-intercepts $x = \pm 1/\sqrt{2}$ and y-intercepts $y = \pm 1$.

   (c) Use the gradient of g to find the normal vector $\vec{n} = \vec{i} + (1/2)\vec{j} + \vec{k}$. Then the plane is
   $$(x - 1/2) + (1/2)(y - 1/2) + (z - 1/4) = 0.$$

2. (a) $2\vec{i} + 4\vec{j}$

   (b) $2\vec{i} + 4\vec{j}$ (Yes, it is the same as (a).)

   (c) $2\vec{i} + 4\vec{j} - \vec{k}$

   (d) Let $\vec{u} = (\vec{i} + \vec{j})/\sqrt{2}$. Then
   $$f_{\vec{u}}(1,3) = \operatorname{grad} f(1,3) \cdot \vec{u} = (2)(1/\sqrt{2}) + (4)(1/\sqrt{2}) = 6/\sqrt{2}.$$

   (e) The quadratic approximation near $(1,3)$ is
   $$Q(x,y) = 1 + 2(x - 1) + 4(y - 3) + (x - 1)^2 - (x - 1)(y - 3) + 2(y - 3)^2,$$
   so we have
   $$
   \begin{aligned}
   f(1.2, 3.3) \approx Q(1.2, 3.3) &= 1 + 2(.2) + 4(.3) + (.2)^2 - (.2)(.3) + 2(.3)^2 \\
   &= 1 + .4 + 1.2 + .04 - .06 + .18 = 2.76
   \end{aligned}
   $$

3. (a) There is one critical point at $(2,2)$. f has a local minimum at $(2,2)$, and the minimum value is $f(2,2) = 8$.

   (b) Let $g(x,y) = x^2 + y^2$. Solving $\operatorname{grad} f = \lambda \operatorname{grad} g$ and $g(x,y) = 18$ yields two points: $(3,3)$, with $\lambda = 1/3$; and $(-3,-3)$, with $\lambda = 5/3$. We find $f(3,3) = 10$, and $f(-3,-3) = 58$, so the (global) maximum of f subject to the given constraint is **58**, and the (global) minimum of f subject to the given constraint is **10**.

   (c) We combine the results of (a) and (b): The (global) maximum of f subject to the given constraint is **58**, and it occurs at $(-3,-3)$. The (global) minimum of f subject to the given constraint is **8**, and it occurs at $(2,2)$.

   (d) The Lagrange multiplier λ gives the rate of change of the maximum value with respect to changes in the constraint constant. We can use this to approximate the change in the maximum value. Recall from (b) that at $(-3,-3)$, we found $\lambda = 5/3$. The *approximate* change in the maximum value is then $\lambda(18.3 - 18) = 0.5$. Thus, the approximate maximum value of f when the constraint equation is $x^2 + y^2 = 18.3$ is **58.5**.

4. (a) Description instead of a sketch: R is the quarter of a disk with radius 3 that is in the first quadrant.

   (b) Remember that in polar coordinates, $dA = r\,d\theta\,dr$, so one of the "r"s in the integrand "belongs to" dA. This means that the function f, expressed in polar coordinates, is r (not r^2). Then, in Cartesian coordinates, f is $\sqrt{x^2 + y^2}$. In Cartesian coordinates, the integral becomes

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2 + y^2}\, dy\, dx.$$

   (c) $\dfrac{9\pi}{2}$

5. (a) $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{x+y} (1 + x + 2z)\, dz\, dy\, dx$

   (b) $\displaystyle\int_0^2 \int_0^{\pi/2} \int_0^{r\cos\theta + r\sin\theta} (1 + r\cos\theta + 2z) r\, dz\, d\theta\, dr$

6. (a) In vector form, the equation of a line is $\vec{r} = \vec{r}_0 + t\vec{v}$, where $\vec{r}_0$ is the position vector of a point in the line, and $\vec{v}$ is a vector in the direction of the line. We already have $\vec{r}_0 = \vec{i} + \vec{j} + \vec{k}$. Let $f(x, y, z) = x^2 + y^2 + 2z^2$. Since the gradient vector of f is perpendicular to the level surface, we can use it for $\vec{v}$. That is, $\vec{v} = \operatorname{grad} f(1, 1, 1) = 2\vec{i} + 2\vec{j} + 4\vec{k}$. Thus the equation of the line is

$$\vec{r} = \vec{i} + \vec{j} + \vec{k} + t(2\vec{i} + 2\vec{j} + 4\vec{k}),$$

   or

$$x = 1 + 2t, \quad y = 1 + 2t, \quad z = 1 + 4t.$$

   (b) We can find the points by first finding the values of t at which the line intersects the surface $x^2 + y^2 + 2z^2 = 4$. Plugging the parametric equations into the equation of the surface, we have

$$(1 + 2t)^2 + (1 + 2t)^2 + 2(1 + 4t)^2 = 4$$
$$40t^2 + 24t + 4 = 4$$
$$t(5t + 3) = 0$$

   so $t = 0$ or $t = -3/5$. At $t = 0$, the parametric equations of the line give the point $(1, 1, 1)$, which is the point we already knew. At $t = -3/5$, the parametric equations of the line give $(-1/5, -1/5, -7/5)$. This is the other point that we want.

7. (a)

$$\frac{dw}{dt} = Q_x(x, y, z)\frac{dx}{dt} + Q_y(x, y, z)\frac{dy}{dt} + Q_z(x, y, z)\frac{dz}{dt}$$
$$= Q_x(f(t), g(t), h(t))f'(t) + Q_y(f(t), g(t), h(t))g'(t) + Q_z(f(t), g(t), h(t))h'(t)$$

(b) Since
$$\operatorname{grad} Q(x, y, z) = Q_x(x, y, z)\vec{i} + Q_y(x, y, z)\vec{j} + Q_z(x, y, z)\vec{k},$$

and
$$\frac{d\vec{r}}{dt} = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k},$$

we have
$$(\operatorname{grad} Q) \cdot \frac{d\vec{r}}{dt} = Q_x(x, y, z)\frac{dx}{dt} + Q_y(x, y, z)\frac{dy}{dt} + Q_z(x, y, z)\frac{dz}{dt}$$
$$= \frac{dw}{dt}.$$