Math 113 – Calculus III **Exam 2 Practice Problems** **Spring 2004**

1. A train is traveling northwest at 10 miles per hour. A person in the train walks at 2 miles per hour from a window on the left side to a window directly across the train on the right side. (*Left* and *right* refer to the sides relative to a person facing the front of the train.)

   Assume that $\vec{i}$ points east, and $\vec{j}$ points north. Express your answers in terms of these unit vectors.

   (a) What is the velocity vector of the train?

   (b) What is the velocity vector of the person relative to the train?

   (c) What is the velocity vector of the person relative to the ground?

   (d) What is the speed of the person relative to the ground?

2. (a) TRUE or FALSE? For any vectors $\vec{v}$ and $\vec{w}$, $(\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w}) = \|\vec{v}\|^2 - \|\vec{w}\|^2$. (Briefly explain.)

   (b) TRUE or FALSE? For any vectors $\vec{v}$ and $\vec{w}$, $\|\vec{v}+\vec{w}\| = \|\vec{v}\| + \|\vec{w}\|$. (Briefly explain.)

3. Let

$$\vec{v} = 3\vec{i} + 2\vec{j} + \vec{k}, \quad \vec{w} = \vec{i} - \vec{j} + \vec{k}, \quad \vec{p} = a\vec{i} - \frac{1}{4}\vec{j} + \frac{1}{2}\vec{k}, \quad \vec{q} = (3-a)\vec{i} - 4\vec{k}.$$

   Find:

   (a) the cosine of the angle between $\vec{v}$ and $\vec{w}$;

   (b) the values of a for which the vectors $\vec{p}$ and $\vec{q}$ perpendicular;

   (c) the values of a for which $\vec{p}$ is a unit vector;

   (d) the vector that is the component of $\vec{v}$ parallel to $\vec{w}$.

4. Given the plane

$$x + y + z = 1,$$

   find the point in the plane that is closest to the point $P = (3, 3, 2)$.

5. Simplify the following expression as much as possible:

$$((\vec{v}\cdot\vec{u})\vec{u})\cdot(\vec{v}\times\vec{w}) - (\vec{w}\times\vec{v})\cdot(\vec{v} - (\vec{u}\cdot\vec{v})\vec{u}).$$

6. Let $P = (1, 1, 1)$, $Q = (1, -3, 0)$ and $R = (2, 2, 2)$.
   (a) Find the equation of the plane that contains the points P, Q, and R.
   (b) Find the area of the triangle formed by the three points.
   (c) Find the distance from the plane found in (a) to the point $(3, 4, 5)$.

7. We say two planes are perpendicular if their normal vectors are perpendicular. Given the following two planes (which are *not* perpendicular):

$$x + 2y + 4z = 1, \qquad -x + y - 2z = 5,$$

find the equation of a plane that is perpendicular to *both* of these planes, and that contains the point $(3, 2, 1)$.

8. Suppose the motion of a particle is given by

$$x = 4\cos t, \qquad y = \sin t.$$

   (a) Describe the motion of the particle, and sketch the curve along which the particle travels.
   (b) Find the velocity and acceleration vectors of the particle.
   (c) Find the times t and the points on the curve where the speed of the particle is greatest.
   (d) Find the times t and the points on the curve where the magnitude of the acceleration is greatest.

9. Find the coordinates of the points where the line

$$x = t, \quad y = 1 + t, \quad z = 5t$$

intersects the surface

$$z = x^2 + y^2.$$

SOLUTIONS

1. (a) The direction "northwest" corresponds to an angle of $3\pi/4$, so the velocity of the train is

$$10\cos(3\pi/4)\vec{i} + 10\sin(3\pi/4)\vec{j} = 10\left(-\frac{\sqrt{2}}{2}\right)\vec{i} + 10\left(\frac{\sqrt{2}}{2}\right)\vec{j} = -5\sqrt{2}\vec{i} + 5\sqrt{2}\vec{j}.$$

(b) Since the train is heading northwest, and the person is walking left to right across the train, the person is facing northeast. So the direction of the person is given by an angle of $\pi/4$, and the velocity of the person relative to the train is

$$2\cos(\pi/4)\vec{i} + 2\sin(\pi/4)\vec{j} = 2\left(\frac{\sqrt{2}}{2}\right)\vec{i} + 2\left(\frac{\sqrt{2}}{2}\right)\vec{j} = \sqrt{2}\vec{i} + \sqrt{2}\vec{j}.$$

(c) This is the sum of the vectors found in (a) and (b):

$$(-5\sqrt{2}\vec{i} + 5\sqrt{2}\vec{j}) + (\sqrt{2}\vec{i} + \sqrt{2}\vec{j}) = -4\sqrt{2}\vec{i} + 6\sqrt{2}\vec{j}.$$

(d) This is the magnitude of the vector found in (c):

$$\sqrt{(-4\sqrt{2})^2 + (6\sqrt{2})^2} = \sqrt{104}.$$

2. (a) TRUE:

$$\begin{aligned}(\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w}) &= \vec{v}\cdot(\vec{v}-\vec{w}) + \vec{w}\cdot(\vec{v}-\vec{w})\\ &= (\vec{v}\cdot\vec{v}) - (\vec{v}\cdot\vec{w}) + (\vec{w}\cdot\vec{v}) - (\vec{w}\cdot\vec{w})\\ &= (\vec{v}\cdot\vec{v}) - (\vec{w}\cdot\vec{w})\\ &= \|\vec{v}\|^2 - \|\vec{w}\|^2.\end{aligned}$$

(Remember that $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$, and $\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$.)

(b) FALSE. For example, take $\vec{v} = \vec{i}$ and $\vec{w} = -\vec{i}$. The $\vec{v}+\vec{w} = \vec{0}$, so $\|\vec{v}+\vec{w}\| = \|\vec{0}\| = 0$, but $\|\vec{v}\| + \|\vec{w}\| = 2$.

3. (a) $\vec{v}\cdot\vec{w} = 2$, $\|\vec{v}\| = \sqrt{14}$, and $\|\vec{w}\| = \sqrt{3}$, so, if θ is the angle between $\vec{v}$ and $\vec{w}$,

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{2}{\sqrt{14}\sqrt{3}}.$$

(b) The vectors $\vec{p}$ and $\vec{q}$ are perpendicular when $\vec{p}\cdot\vec{q} = 0$. Thus we want

$$a(3-a) + 0 - 2 = 0 \implies a^2 - 3a + 2 = 0 \implies a = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2}.$$

The vectors are perpendicular if $a = 1$ or $a = 2$.

(c) $\vec{p}$ is a unit vector if $\|\vec{p}\| = 1$. Thus we want

$$\sqrt{a^2 + \frac{1}{16} + \frac{1}{4}} = 1 \implies a^2 + \frac{5}{16} = 1 \implies a^2 = \frac{11}{16} \implies a = \pm\frac{\sqrt{11}}{4}.$$

$\vec{p}$ is a unit vector if $a = -\sqrt{11}/4$ or $a = \sqrt{11}/4$.

(d) Let $\vec{u} = \dfrac{\vec{w}}{\|\vec{w}\|} = \dfrac{1}{\sqrt{3}}(\vec{i} - \vec{j} + \vec{k})$. Then

$$\vec{v}_{\text{par}} = (\vec{v}\cdot\vec{u})\vec{u} = \left(\frac{(3)(1) + (2)(-1) + (1)(1)}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}(\vec{i} - \vec{j} + \vec{k})\right) = \frac{2}{3}(\vec{i} - \vec{j} + \vec{k})$$

4. Let $C = (x_c, y_c, z_c)$ be the point in the plane closest to P. We will find the vector $\overrightarrow{CP}$, from which we can find C.

Let $P_0 = (0, 0, 1)$; this is a point in the plane. A vector normal to the plane is $\vec{n} = \vec{i} + \vec{j} + \vec{k}$, and a unit vector in the same direction as $\vec{n}$ is

$$\vec{u} = \frac{\vec{n}}{\|\vec{n}\|} = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k}).$$

Let $\vec{v} = \overrightarrow{P_0P} = 3\vec{i} + 3\vec{j} + \vec{k}$. Then $\overrightarrow{CP}$ is the component of $\vec{v}$ parallel to $\vec{u}$ (i.e. $\overrightarrow{CP}$ is the projection of $\vec{v}$ on $\vec{u}$). Thus

$$\overrightarrow{CP} = \vec{v}_{\text{par}} = (\vec{v}\cdot\vec{u})\vec{u} = \frac{7}{3}(\vec{i} + \vec{j} + \vec{k}) = \frac{7}{3}\vec{i} + \frac{7}{3}\vec{j} + \frac{7}{3}\vec{k}.$$

Since $\overrightarrow{CP}$ is the displacement vector from C to P, we also know

$$\overrightarrow{CP} = (3 - x_c)\vec{i} + (3 - y_c)\vec{j} + (2 - z_c)\vec{k}.$$

Thus

$$3 - x_c = \frac{7}{3} \implies x_c = \frac{2}{3}, \quad 3 - y_c = \frac{7}{3} \implies y_c = \frac{2}{3}, \quad 2 - z_c = \frac{7}{3} \implies z_c = -\frac{1}{3}.$$

so $C = \left(\dfrac{2}{3}, \dfrac{2}{3}, -\dfrac{1}{3}\right)$.

5.

$$\begin{aligned}
&((\vec{v}\cdot\vec{u})\vec{u})\cdot(\vec{v}\times\vec{w}) - (\vec{w}\times\vec{v})\cdot(\vec{v} - (\vec{u}\cdot\vec{v})\vec{u}) \\
&\qquad = ((\vec{v}\cdot\vec{u})\vec{u})\cdot(\vec{v}\times\vec{w}) + (\vec{v}\times\vec{w})\cdot(\vec{v} - (\vec{v}\cdot\vec{u})\vec{u}) \\
&\qquad = [(\vec{v}\cdot\vec{u})\vec{u} + \vec{v} - (\vec{v}\cdot\vec{u})\vec{u}]\cdot(\vec{v}\times\vec{w}) \\
&\qquad = \vec{v}\cdot(\vec{v}\times\vec{w}) \\
&\qquad = 0.
\end{aligned}$$

6. (a) We have a point in the plane (in fact, we have three). All we need is a normal vector. This is given by

$$\begin{aligned}
\vec{n} &= \overrightarrow{PQ} \times \overrightarrow{PR} \\
&= \left((1-1)\vec{i} + (-3-1)\vec{j} + (0-1)\vec{k}\right) \times \left((2-1)\vec{i} + (2-1)\vec{j} + (2-1)\vec{k}\right) \\
&= \left(-4\vec{j} - \vec{k}\right) \times \left(\vec{i} + \vec{j} + \vec{k}\right) \\
&= -3\vec{i} - \vec{j} + 4\vec{k}.
\end{aligned}$$

By using $\vec{n}$ and the point P, we find the equation of the plane to be

$$-3(x-1) - (y-1) + 4(z-1) = 0 \quad \text{or} \quad -3x - y + 4z = 0.$$

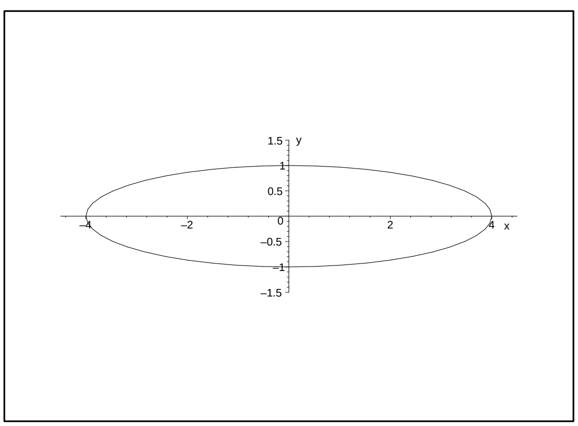
(b) $\frac{\|\vec{n}\|}{2} = \frac{\sqrt{9+1+16}}{2} = \frac{\sqrt{26}}{2}$

(c) Let A be the point $(3,4,5)$. Let $\vec{u} = \frac{\vec{n}}{\|\vec{n}\|} = \frac{1}{\sqrt{26}}\left(-3\vec{i} - \vec{j} + 4\vec{k}\right)$. Then $d = |\overrightarrow{PA} \cdot \vec{u}| = \left|\frac{1}{\sqrt{26}}\left(2\vec{i} + 3\vec{j} + 4\vec{k}\right) \cdot \left(-3\vec{i} - \vec{j} + 4\vec{k}\right)\right| = \frac{7}{\sqrt{26}}$.

7. Normal vectors for the two given planes are $\vec{n}_1 = \vec{i} + 2\vec{j} + 4\vec{k}$ and $\vec{n}_2 = -\vec{i} + \vec{j} - 2\vec{k}$, respectively. Then $\vec{n}_3 = \vec{n}_1 \times \vec{n}_2 = -8\vec{i} - 2\vec{j} + 3\vec{k}$ is perpendicular to both $\vec{n}_1$ and $\vec{n}_2$, and therefore a plane with normal vector $\vec{n}_3$ will be perpendicular to the given planes. We are told that $(3,2,1)$ is a point in the plane, so the equation of the plane is

$$-8(x-3) - 2(y-2) + 3(z-1) = 0.$$

8. (a) Note that $(x/4)^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so the path of the particle is the *ellipse* $(x/4)^2 + y^2 = 1$. The motion is counter-clockwise around the ellipse.



(b) The path is $\vec{r}(t) = (4\cos t)\vec{i} + (\sin t)\vec{j}$, so the velocity is

$$\vec{v}(t) = \vec{r}\,'(t) = (-4\sin t)\vec{i} + (\cos t)\vec{j}$$

and the acceleration is

$$\vec{a}(t) = \vec{r}\,''(t) = (-4\cos t)\vec{i} + (-\sin t)\vec{j}.$$

(c) The speed is

$$v = \|\vec{v}(t)\| = \sqrt{(-4\sin t)^2 + (\cos t)^2} = \sqrt{16\sin^2 t + \cos^2 t}.$$

This has a maximum or minimum when $dv/dt = 0$. Now

$$\frac{dv}{dt} = \frac{15\sin t\cos t}{\sqrt{16\sin^2 t + \cos^2 t}},$$

so $dv/dt = 0$ when $\sin t = 0$ or $\cos t = 0$. This means $t = n\pi$ or $t = \pi/2 + n\pi$, where n is any integer. When $t = n\pi$, $v = 1$, and when $t = \pi/2 + n\pi$, $v = 4$. Thus the maximum speed is $v = 4$. It occurs when $t = \pi/2 + n\pi$. When n is even, the point in the plane where this occurs is $(0, 1)$, and when n is odd, the point is $(0, -1)$.

(d) The magnitude of the acceleration is

$$a = \|\vec{a}(t)\| = \sqrt{(-4\cos t)^2 + (-\sin t)^2} = \sqrt{16\cos^2 t + \sin^2 t}.$$

This has a maximum or minimum when $da/dt = 0$. Now

$$\frac{da}{dt} = \frac{-15\sin t\cos t}{\sqrt{16\cos^2 t + \sin^2 t}},$$

so $da/dt = 0$ when $\sin t = 0$ or $\cos t = 0$. This means $t = n\pi$ or $t = \pi/2 + n\pi$, where n is any integer. When $t = n\pi$, $a = 4$, and when $t = \pi/2 + n\pi$, $a = 1$. Thus the maximum magnitude of the acceleration is $a = 4$. It occurs when $t = n\pi$. When n is even, the point in the plane where this occurs is $(0, 4)$, and when n is odd, the point is $(0, -4)$.

9. We have $x = t$, $y = 1 + t$ and $z = 5t$, and we want $z = x^2 + y^2$, so we must solve

$$5t = t^2 + (1+t)^2 \implies 2t^2 - 3t + 1 = 0 \implies t = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4}.$$

So the line intersects the surface when $t = 1/2$ and when $t = 1$. When $t = 1/2$, the point of intersection is $(1/2, 3/2, 5/2)$, and when $t = 1$, the point of intersection is $(1, 2, 5)$.