Math 113 – Calculus III **Exam 3 Practice Problems** **Spring 2004**

1. Let
$$g(x,y,z) = e^{-(x+y)^2} + z^2(x+y).$$
Suppose that a piece of fruit is sitting on a table in a room, and at each point (x,y,z) in the space within the room, $g(x,y,z)$ gives the strength of the odor of the fruit. Furthermore, suppose that a certain bug always flies in the direction in which the fruit odor increases fastest. Suppose also that the bug always flies with a *speed* of 2 feet/second.

   What is the velocity vector of the bug when it is at the position $(2,-2,1)$?

2. The path of a particle in space is given by the functions $x(t) = 2t$, $y(t) = \cos(t)$, and $z(t) = \sin(t)$. Suppose the temperature in this space is given by a function $H(x,y,z)$. Find $\frac{dH}{dt}$, the rate of change of the temperature at the particle's position. (Since the actual function $H(x,y,z)$ is not given, your answer will be in terms of derivatives of H.)

3. Let
$$f(x,y) = x^3 - xy + \cos(\pi(x+y)).$$
   (a) Find a vector normal to the level curve $f(x,y) = 1$ at the point where $x = 1$, $y = 1$.
   (b) Find the equation of the line tangent to the level curve $f(x,y) = 1$ at the point where $x = 1$, $y = 1$.
   (c) Find a vector normal to the graph $z = f(x,y)$ at the point $x = 1$, $y = 1$.
   (d) Find the equation of the plane tangent to the graph $z = f(x,y)$ at the point $x = 1$, $y = 1$.

4. Suppose f is a differentiable function such that
$$f(1,3) = 1, \quad f_x(1,3) = 2, \quad f_y(1,3) = 4,$$
$$f_{xx}(1,3) = 2, \quad f_{xy}(1,3) = -1, \quad \text{and} \quad f_{yy}(1,3) = 4.$$
   (a) Find $\text{grad} f(1,3)$.
   (b) Find a vector in the plane that is perpendicular to the contour line $f(x,y) = 1$ at the point $(1,3)$.
   (c) Find a vector that is perpendicular to the surface $z = f(x,y)$ (i.e. the graph of f) at the point $(1,3,1)$.
   (d) At the point $(1,3)$, what is the rate of change of f in the direction $\vec{i} + \vec{j}$?
   (e) Use a quadratic approximation to estimate $f(1.2, 3.3)$.

5. We say that a line in 3-space is *normal to a surface* at a point of intersection if the line is normal to (i.e. perpendicular to) the tangent plane of the surface at that point.

   Let S be the surface defined by

$$x^2 + y^2 + 2z^2 = 4.$$

   (a) Find the parametric equations of the line that is normal to the surface S at the point $(1, 1, 1)$.

   (b) The line found in (a) will intersect the surface S at two points. One of them is $(1, 1, 1)$, by construction. Find the other point of intersection.

6. Let

$$f(x, y) = (x - y)^3 + 2xy + x^2 - y.$$

   (a) Find the function $L(x, y)$ that gives the linear approximation of f near the point $(1, 2)$.

   (b) Find the function $Q(x, y)$ that gives the quadratic approximation of f near the point $(1, 2)$

7. For each of the following functions, determine the set of points where the function is *not* differentiable. *Briefly* explain how you know it is not differentiable; use a picture if it helps.

   (You do not have to *prove* that it is not differentiable; just identify the set of points based on your understanding of what differentiable means.)

   (a) $f(x, y) = \left|x^2 + y^2 - 1\right|$

   (b) $f(x, y) = (x^2 + y^2)^{1/4}$

   (c) $f(x, y) = e^{-x^2+y}$

   (d) $f(x, y) = \dfrac{x^3 - xy + 1}{x^2 - y^2}$

8. Suppose $w = Q(x, y, z)$, where Q is a differentiable function. Next suppose that $x = f(t)$, $y = g(t)$ and $z = h(t)$.

   (a) Use the chain rule to find an expression for $\frac{dw}{dt}$ in terms of Q, f, g, h and their derivatives (e.g. Q_x, f', etc.).

   (b) Show that the expression in (a) may be written as

$$\frac{dw}{dt} = (\operatorname{grad} Q(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt},$$

   where $\cdot$ is the dot product, $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ is the vector form of the parameterized curve $x = f(t)$, $y = g(t)$, and $z = h(t)$, and, if $\vec{r} = a\vec{i} + b\vec{j} + c\vec{k}$, $Q(\vec{r})$ means $Q(a, b, c)$.

9. (a) Give an example of a function $f(x, y)$ for which $(0, 0)$ is a local minimum but for which the second derivative test fails to determine this classification.

   (b) Give an example of a function that has a global maximum at the point $(3, 2)$.

   (c) Suppose f is a differentiable function, and

$$f(0,0) = 0, \quad f_x(0,0) = 0, \quad f_y(0,0) = 0,$$
$$f_{xx}(0,0) = 4, \quad f_{yy}(0,0) = 0, \quad \text{and} \quad f_{xy}(0,0) = 3$$

      i. If possible, classify the origin as a local maximum, local minimum, or saddle point.

      ii. Sketch a possible contour diagram of f near the origin. Be sure to label your contour lines.

   (d) Let R be the region in the plane where $x^2 + y^2 \leq 4$. Either give an example of a function defined on R which has no global maximum, or explain why such a function is not possible.

10. Find the critical points of

$$f(x, y) = xy^2 + x^3 - 2xy,$$

and classify each critical points as either a local maximum, local minimum, or saddle point (if possible).

Solutions

1. Since the bug flies in the direction in which the fruit odor increases fastest, it flies in the direction of grad g. The gradient of g is

$$\operatorname{grad} g(x,y,z) = \left(-2(x+y)e^{-(x+y)^2} + z^2\right)\vec{i} + \left(-2(x+y)e^{-(x+y)^2} + z^2\right)\vec{j}$$
$$+ \left(2z(x+y)\right)\vec{k},$$

and

$$\operatorname{grad} g(2,-2,1) = \vec{i} + \vec{j}.$$

The bug always has a speed of 2, so the velocity vector must have a magnitude of 2. A vector with magnitude 2 and in the same direction as the gradient is

$$2\frac{\operatorname{grad} g(2,-2,1)}{\|\operatorname{grad} g(2,-2,1)\|} = \frac{2}{\sqrt{2}}(\vec{i} + \vec{j}).$$

2. By the chain rule,

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} + \frac{\partial H}{\partial z}\frac{dz}{dt} = 2\frac{\partial H}{\partial x} - \sin t\frac{\partial H}{\partial y} + \cos t\frac{\partial H}{\partial z}$$

3. (a) The gradient of f is normal to the level curve at each point. We find

$$\operatorname{grad} f(x,y) = (3x^2 - y - \pi\sin(\pi(x+y)))\vec{i} + (-x - \pi\sin(\pi(x+y)))\vec{j},$$

and

$$\operatorname{grad} f(1,1) = 2\vec{i} - \vec{j},$$

so one possible answer is $2\vec{i} - \vec{j}$.

(b) The line is

$$2(x-1) - (y-1) = 0, \quad \text{or} \quad 2x - y = 1.$$

(c) The graph is the level surface $g(x,y,z) = 0$ of the function $g(x,y,z) = f(x,y) - z$. The gradient of g is normal to the level surface at each point. We have $\operatorname{grad} g(x,y,z) = \operatorname{grad} f(x,y) - \vec{k}$. The point on the surface where $(x,y) = (1,1)$ is $(1,1,f(1,1)) = (1,1,1)$. A vector normal to the graph at $(1,1,1)$ is

$$\operatorname{grad} g(1,1,1) = \operatorname{grad} f(1,1) - \vec{k} = 2\vec{i} - \vec{j} - \vec{k}.$$

(d) The plane is $\quad 2(x-1) - (y-1) - (z-1) = 0, \quad$ or $\quad 2x - y - z = 0.$

4. (a) $\operatorname{grad} f(1,3) = f_x(1,3)\vec{i} + f_y(1,3)\vec{j} = 2\vec{i} + 4\vec{j}$

(b) $2\vec{i} + 4\vec{i}$ (from (a); the gradient vector at a point is perpendicular to the contour line through that point)

(c) The graph is the level surface $g(x,y,z)=0$ of the function $g(x,y,z)=f(x,y)-z$. The gradient of g is normal to the level surface at each point. We have $\text{grad}\, g(x,y,z)=\text{grad}\, f(x,y)-\vec{k}$. The point on the surface where $(x,y)=(1,3)$ is $(1,3,f(1,3))=(1,3,1)$. A vector normal to the graph at $(1,3,1)$ is

$$\text{grad}\, g(1,3,1)=\text{grad}\, f(1,3)-\vec{k}=2\vec{i}+4\vec{j}-\vec{k}.$$

(d) $\vec{u}=(\vec{i}+\vec{j})/\sqrt{2}$ is a unit vector in the direction of $\vec{i}+\vec{j}$. The rate of change of f in this direction is $f_{\vec{u}}(1,3)=\text{grad} f(1,3)\cdot\vec{u}=(2\vec{i}+4\vec{j})\cdot(\vec{i}+\vec{j})/\sqrt{2}=6/\sqrt{2}=3\sqrt{2}$.

(e) Near $(1,3)$, we have

$$\begin{aligned} f(x,y)\approx f(1,3)+&f_x(1,3)(x-1)+f_y(1,3)(y-3)+\\ &\frac{f_{xx}(1,3)}{2}(x-1)^2+f_{xy}(1,3)(x-1)(y-3)+\frac{f_{yy}(1,3)}{2}(y-3)^2. \end{aligned}$$

So

$$\begin{aligned} f(1.2,3.3)&\approx 1+(2)(0.2)+(4)(0.3)+(2/2)(0.2)^2+(-1)(0.2)(0.3)+(4/2)(0.3)^2\\ &= 2.76. \end{aligned}$$

5. (a) In vector form, the equation of a line is $\vec{r}=\vec{r}_0+t\vec{v}$, where $\vec{r}_0$ is the position vector of a point in the line, and $\vec{v}$ is a vector in the direction of the line. We already have $\vec{r}_0=\vec{i}+\vec{j}+\vec{k}$. Let $f(x,y,z)=x^2+y^2+2z^2$. Since the gradient vector of f is perpendicular to the level surface, we can use it for $\vec{v}$. That is, $\vec{v}=\text{grad}\, f(1,1,1)=2\vec{i}+2\vec{j}+4\vec{k}$. Thus the equation of the line is

$$\vec{r}=\vec{i}+\vec{j}+\vec{k}+t(2\vec{i}+2\vec{j}+4\vec{k}),$$

or

$$x=1+2t,\quad y=1+2t,\quad z=1+4t.$$

(b) We can find the points by first finding the values of t at which the line intersects the surface $x^2+y^2+2z^2=4$. Plugging the parametric equations into the equation of the surface, we have

$$\begin{aligned} (1+2t)^2+(1+2t)^2+2(1+4t)^2&=4\\ 40t^2+24t+4&=4\\ t(5t+3)&=0 \end{aligned}$$

so $t=0$ or $t=-3/5$. At $t=0$, the parametric equations of the line give the point $(1,1,1)$, which is the point we already knew. At $t=-3/5$, the parametric equations of the line give $(-1/5,-1/5,-7/5)$. This is the other point that we want.

6. (a) First get the numbers:
$f(1,2) = -1+4+1-2 = 2$,
$f_x(x,y) = 3(x-y)^2 + 2y + 2x, \qquad f_x(1,2) = 3+4+2 = 9$,
$f_y(x,y) = -3(x-y)^2 + 2x - 1, \qquad f_y(1,2) = -3+2-1 = -2$.
Then

$$\begin{aligned} L(x,y) &= f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) \\ &= 2 + 9(x-1) - 2(y-2). \end{aligned}$$

(b) We need some more numbers:
$f_{xx}(x,y) = 6(x-y) + 2, \qquad f_{xx}(1,2) = -6+2 = -4$,
$f_{xy}(x,y) = -6(x-y) + 2, \qquad f_{xy}(1,2) = 6+2 = 8$,
$f_{yy}(x,y) = 6(x-y), \qquad f_{yy}(1,2) = -6$.
Then

$$\begin{aligned} Q(x,y) &= L(x,y) + \frac{f_{xx}(1,2)}{2}(x-1)^2 + f_{xy}(1,2)(x-1)(y-2) + \frac{f_{yy}(1,2)}{2}(y-2)^2 \\ &= 2 + 9(x-1) - 2(y-2) - 2(x-1)^2 + 8(x-1)(y-2) - 3(y-2)^2. \end{aligned}$$

7. (a) This function is not differentiable on the circle $x^2 + y^2 = 1$. The graph has a "corner" at these points.

(b) This function is not differentiable at the origin. Consider the cross section $y = 0$: $f(x,0) = (x^2)^{1/4} = \sqrt{|x|}$. The graph has a cusp (i.e. a point) at $x = 0$.

(c) This function is the composition of polynomials and the exponential function, so it is differentiable everywhere.

(d) This function is not differentiable at points where the denominator is zero; that is, where $x^2 = y^2$. This gives the lines $y = x$ and $y = -x$.

8. (a)

$$\begin{aligned} \frac{dw}{dt} &= Q_x(x,y,z)\frac{dx}{dt} + Q_y(x,y,z)\frac{dy}{dt} + Q_z(x,y,z)\frac{dz}{dt} \\ &= Q_x(f(t),g(t),h(t))f'(t) + Q_y(f(t),g(t),h(t))g'(t) + Q_z(f(t),g(t),h(t))h'(t) \\ &= Q_x(\vec{r}(t))f'(t) + Q_y(\vec{r}(t))g'(t) + Q_z(\vec{r}(t))h'(t) \end{aligned}$$

(b) Since

$$\operatorname{grad} Q(x,y,z) = Q_x(x,y,z)\vec{i} + Q_y(x,y,z)\vec{j} + Q_z(x,y,z)\vec{k},$$

we have

$$\operatorname{grad} Q(\vec{r}(t)) = Q_x(\vec{r}(t))\vec{i} + Q_y(\vec{r}(t))\vec{j} + Q_z(\vec{r}(t))\vec{k},$$

Also,

$$\frac{d\vec{r}}{dt} = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$$

so we have

$$\frac{dw}{dt} = Q_x(\vec{r}(t))f'(t) + Q_y(\vec{r}(t))g'(t) + Q_z(\vec{r}(t))h'(t)$$
$$= (\operatorname{grad} Q(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}$$

9. (a) One possibility is $f(x, y) = x^4 + y^4$. (See your class notes, or see Example 7 on page 708 of the text.)

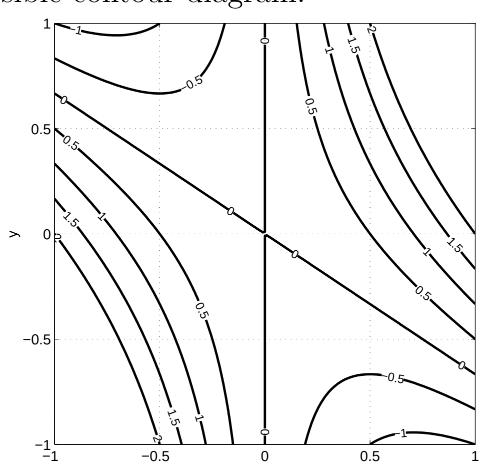
   (b) A quadratic function seems like the simplest choice. One possible solution is $f(x, y) = -(x-3)^2 - (y-2)^2$.

   (c) i. Since $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$, we know $(0, 0)$ is a critical point of f. We find $D = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = -9 < 0$, so $(0, 0)$ is a *saddle point.*

   ii. The second derivative test tells us that the graph is a saddle. (Since the second derivative test is based on the degree 2 Taylor approximation, this is not a "monkey saddle" or something more complicated.) Since $f(0, 0) = 0$, we know two contour lines (where $f(x, y) = 0$) will meet at $(0, 0)$. One of these must be along the y axis, since $f_{yy}(0, 0) = 0$. (If there was not a contour line tangent to the y axis at $(0, 0)$, we would not have $f_{yy}(0, 0) = 0$.) Now, since $f_{xy}(0, 0) = 3$, f_x will increase along the positive y axis. This means that $f(x, y) > 0$ just to the right of the positive y axis, and $f(x, y) < 0$ just to the left of the positive y axis.

   Since $f_{xx}(0, 0) = 4 > 0$, $f(x, 0) > 0$ along the x axis. Now consider the values of f in the second quadrant ($x < 0$, $y > 0$). Near the x axis, $f(x, y) > 0$, but near the y axis, $f(x, y) < 0$. So the contour line $f(x, y) = 0$ must cut through the second quadrant. (It must also, therefore, cut through the fourth quadrant.)

   Here is a possible contour diagram:



   This happens to be the function $f(x, y) = 2x^2 + 3xy$ (but you did not have to find an actual function to sketch a qualitatively correct contour diagram).

(d) Here is one possibility:

$$f(x,y) = \begin{cases} \frac{1}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

This function has no maximum value on R, because $f(x,y)$ becomes arbitrarily large as $(x,y) \to (0,0)$. Note that this function *is not continuous* on R, so this example does not contradict the Extreme Value Theorem (Theorem 15.1).

10. First, we note that this function is a polynomial in x and y, so it is differentiable at all points. Therefore the only critical points are the points where $\operatorname{grad} f(x,y) = \vec{0}$. We have

$$f_x(x,y) = y^2 + 3x^2 - 2y, \quad \text{and} \quad f_y(x,y) = 2xy - 2x.$$

We must solve

$$\boxed{1} \quad y^2 + 3x^2 - 2y = 0 \quad \text{and} \quad \boxed{2} \quad 2xy - 2x = 0.$$

The second equation looks simpler than the first, so we'll start there. We have $2xy - 2y = 2x(y-1) = 0$, so either $x = 0$ or $y = 1$.

If $x = 0$, $\boxed{1}$ implies $y^2 - 2y = 0$ or $y(y-2) = 0$, so $y = 0$ or $y = 2$. Thus two critical points are $(0,0)$ and $(0,2)$.

If $y = 1$, $\boxed{1}$ implies $3x^2 = 1$ or $x = \pm 1/\sqrt{3}$. Thus two more critical points are $(1/\sqrt{3}, 1)$ and $(-1/\sqrt{3}, 1)$.

The critical points are

$$(0,0), \quad (0,2), \quad \left(\frac{1}{\sqrt{3}}, 1\right), \quad \text{and} \quad \left(\frac{-1}{\sqrt{3}}, 1\right).$$

We will need

$$f_{xx}(x,y) = 6x, \quad f_{yy}(x,y) = 2x, \quad \text{and} \quad f_{xy}(x,y) = 2y - 2.$$

The discriminant at a critical point (x_0, y_0) is then

$$D(x_0, y_0) = (6x_0)(2x_0) - (2y_0 - 2)^2 = 12x_0^2 - 4(y_0 - 1)^2.$$

Now check each critical point:

$D(0,0) = -4 < 0$, so $(0,0)$ is a saddle point.

$D(0,2) = -4 < 0$, so $(0,2)$ is a saddle point.

$D(1/\sqrt{3}, 1) = 4 > 0$, and $f_{xx}(1/\sqrt{3}, 1) = 6/\sqrt{3} > 0$, so f has a local minimum at $(1/\sqrt{3}, 1)$.

$D(-1/\sqrt{3}, 1) = 4 > 0$, and $f_{xx}(-1/\sqrt{3}, 1) = -6/\sqrt{3} < 0$, so f has a local maximum at $(1/\sqrt{3}, 1)$.