**Math 113 – Calculus III** **Exam 4 Practice Problems** **Spring 2004**

1. The *energy* stored in an idealized stretched spring is proportional to the square of the length of the string.

   Suppose three springs lie in the $xy$ plane. One end of the first spring is fixed at $(1, 0)$, one end of the second spring is fixed at $(0, 0)$ and one end of the third spring is fixed at $(0, 1)$. The other ends of the three springs are connected together. This arrangement of springs will have an *equilibrium* at the point where the total energy is a minimum. The total energy is just the sum of the energy of each spring: $E = E_1 + E_2 + E_3 = k_1 s_1^2 + k_2 s_2^2 + k_3 s_3^2$, where $s_i$ is the length of spring $i$, and the proportionality constants $k_i$ are all positive.

   Let $(x, y)$ be the coordinates of the point where the springs are connected together. Find the equilibrium point of the springs.

2. Let
$$f(x, y) = x^2 - 4x + y^2 - 4y + 16.$$

   (a) Find and classify the critical points of $f$.

   (b) Find the maximum and minimum values of $f$ subject to the constraint
$$x^2 + y^2 = 18$$

   (c) Find the maximum and minimum values of $f$ subject to the constraint
$$x^2 + y^2 \leq 18$$

   (d) Approximate the maximum value of $f$ subject to the constraint
$$x^2 + y^2 = 18.3$$

   (Explain your answer in terms of Lagrange multipliers.)

3. For each of the following, sketch the region of integration, *and* rewrite the integral with the order of integration reversed. Clearly label the boundaries of the region with an equation that defines the boundary curve, and use shading (or cross-hatches) to indicate the region.

   (a) $\displaystyle\int_0^3 \int_0^{9-y^2} f(x, y)\, dx\, dy$

   (b) $\displaystyle\int_{-1}^1 \int_{x^3}^1 f(x, y)\, dy\, dx$

4. For each of the following, sketch the region of integration, *and* evaluate the integral.

   (a) $\displaystyle\int_0^{2\sqrt{2}} \int_x^{\sqrt{16-x^2}} 3x \, dy \, dx$

   (b) $\displaystyle\int_0^1 \int_{-y}^{y} (x^2 + 2y) \, dx \, dy$

5. Consider the solid region inside the cylinder $x^2 + y^2 = a^2$, above the plane $z = 0$ and below the plane $z = a - x$. Let $f(x, y, z) = 5 + z + x^2$ be the density of a substance in this region.

   Set up (*but do not evaluate*) an iterated integral that gives the total amount of the substance in the region.

## Solutions

1. We have

$$s_1^2 = (1-x)^2 + y^2, \quad s_2^2 = x^2 + y^2, \quad s_3^2 = x^2 + (1-y)^2,$$

and

$$E(x,y) = k_1((1-x)^2 + y^2) + k_2(x^2 + y^2) + k_3(x^2 + (1-y)^2).$$

We want to find the minimum of $E$. We have

$$\begin{aligned} E_x(x,y) &= -2k_1(1-x) + 2k_2 x + 2k_3 x = 2(k_1+k_2+k_3)x - 2k_1, \\ E_y(x,y) &= 2k_1 y + 2k_2 y - 2k_3(1-y) = 2(k_1+k_2+k_3)y - 2k_3. \end{aligned}$$

The critical points are given by the solutions to

$$2(k_1+k_2+k_3)x - 2k_1 = 0, \quad \text{and} \quad 2(k_1+k_2+k_3)y - 2k_3 = 0,$$

and the only solution is

$$x = \frac{k_1}{k_1+k_2+k_3}, \quad y = \frac{k_3}{k_1+k_2+k_3}.$$

To verify that this is, in fact, the global minimum of $E(x,y)$, we can simply observe that $E$ is a quadratic polynomial, and it is the sum of squared terms that all have positive coefficients. Therefore the critical point must be a global minimum point.

It is also easy to check this with the second derivative test. We have

$$E_{xx}(x,y) = 2(k_1+k_2+k_3), \quad E_{yy} = 2(k_1+k_2+k_3), \quad E_{xy} = 0.$$

Then

$$D = 4(k_1+k_2+k_3)^2 > 0,$$

and $E_{xx}(x,y) > 0$ since each proportionality constant $k_i$ is positive. Therefore the second derivative test verifies that the critical point is a local minimum.

2. (a) There is one critical point at $(2,2)$. $f$ has a local minimum at $(2,2)$, and the minimum value is $f(2,2) = 8$.

   (b) Let $g(x,y) = x^2 + y^2$. Solving $\operatorname{grad} f = \lambda \operatorname{grad} g$ and $g(x,y) = 18$ yields two points: $(3,3)$, with $\lambda = 1/3$; and $(-3,-3)$, with $\lambda = 5/3$. We find $f(3,3) = 10$, and $f(-3,-3) = 58$, so the (global) maximum of $f$ subject to the given constraint is **58**, and the (global) minimum of $f$ subject to the given constraint is **10**.

   (c) We combine the results of (a) and (b): The (global) maximum of $f$ subject to the given constraint is **58**, and it occurs at $(-3,-3)$. The (global) minimum of $f$ subject to the given constraint is **8**, and it occurs at $(2,2)$.

(d) The Lagrange multiplier $\lambda$ gives the rate of change of the maximum value with respect to changes in the constraint constant. We can use this to approximate the change in the maximum value. Recall from (b) that at $(-3, -3)$, we found $\lambda = 5/3$. The *approximate* change in the maximum value is then $\lambda(18.3 - 18) = 0.5$. Thus, the approximate maximum value of $f$ when the constraint equation is $x^2 + y^2 = 18.3$ is **58.5**.

3. (The regions will be shown in class.)

(a) $\displaystyle\int_0^9 \int_0^{\sqrt{9-x}} f(x, y)\, dy\, dx$

(b) $\displaystyle\int_{-1}^1 \int_{-1}^{y^{1/3}} f(x, y)\, dx\, dy$

4. (The regions will be shown in class.)

(a)

$$\begin{aligned}
\int_0^{2\sqrt{2}} \int_x^{\sqrt{16-x^2}} 3x\, dy\, dx &= \int_0^{2\sqrt{2}} 3xy\Big|_x^{\sqrt{16-x^2}}\, dx \\
&= \int_0^{2\sqrt{2}} 3x\sqrt{16-x^2} - 3x^2\, dx \\
&= -(16-x^2)^{3/2} - x^3\Big|_0^{2\sqrt{2}} \\
&= -32\sqrt{2} + 64
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^1 \int_{-y}^{y} (x^2 + 2y)\, dx\, dy &= \int_0^1 \frac{x^3}{3} + 2xy\Big|_{-y}^{y}\, dy \\
&= \int_0^1 \frac{2y^3}{3} + 4y^2\, dy \\
&= \frac{y^4}{6} + \frac{4y^3}{3}\Big|_0^1 \\
&= \frac{3}{2}
\end{aligned}$$

5. $\displaystyle\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{a-x} (5 + z + x^2)\, dz\, dy\, dx.$