Math 213 – Calculus III **Practice Exam 2 Solutions** **Fall 2002**

1. Let
$$g(x,y,z) = e^{-(x+y)^2} + z^2(x+y).$$

   (a) What is the instantaneous rate of change of g at the point $(2,-2,1)$ in the direction of the origin?

   We want the directional derivative of g at $(2,-2,1)$ in the direction of the origin. A vector in this direction is $-2\vec{i}+2\vec{j}-\vec{k}$, and a unit vector in this direction is $\vec{u} = \frac{1}{\sqrt{9}}(-2\vec{i}+2\vec{j}-\vec{k}) = \left(-\frac{2}{3}\vec{i}+\frac{2}{3}\vec{j}-\frac{1}{3}\vec{k}\right)$. The gradient of g is
$$\operatorname{grad} g(x,y,z) = \left(-2(x+y)e^{-(x+y)^2}+z^2\right)\vec{i} + \left(-2(x+y)e^{-(x+y)^2}+z^2\right)\vec{j} + (2z(x+y))\,\vec{k},$$
and in particular
$$\operatorname{grad} g(2,-2,1) = \vec{i}+\vec{j}.$$
Then the instantaneous rate of change of g in the direction $\vec{u}$ at the point $(2,-2,1)$ is
$$g_{\vec{u}}(2,-2,1) = \operatorname{grad} g(2,-2,1)\cdot\vec{u} = \left(\vec{i}+\vec{j}\right)\cdot\left(-\frac{2}{3}\vec{i}+\frac{2}{3}\vec{j}-\frac{1}{3}\vec{k}\right) = 0.$$

   (b) Suppose that a piece of fruit is sitting on a table in a room, and at each point (x,y,z) in the space within the room, $g(x,y,z)$ gives the strength of the odor of the fruit. Furthermore, suppose that a certain bug always flies in the direction in which the fruit odor increases fastest. Suppose also that the bug always flies with a *speed* of 2 feet/second. What is the velocity vector of the bug when it is at the position $(2,-2,1)$?

   Since the bug flies in the direction in which the fruit odor increases fastest, it flies in the direction of $\operatorname{grad} g$. It always has a speed of 2, so the velocity vector at $(2,-2,1)$ is
$$2\frac{\operatorname{grad} g(2,-2,1)}{\|\operatorname{grad} g(2,-2,1)\|} = \frac{2}{\sqrt{2}}(\vec{i}+\vec{j}).$$

2. The path of a particle in space is given by the functions $x(t)=2t$, $y(t)=\cos(t)$, and $z(t)=\sin(t)$. Suppose the temperature in this space is given by a function $H(x,y,z)$.

   (a) Find $\frac{dH}{dt}$, the rate of change of the temperature at the particle's position. (Since the actual function $H(x,y,z)$ is not given, your answer will be in terms of derivatives of H.)

   $\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} + \frac{\partial H}{\partial z}\frac{dz}{dt} = 2\frac{\partial H}{\partial x} - \sin t\frac{\partial H}{\partial y} + \cos t\frac{\partial H}{\partial z}$

   (b) Suppose we know that at all points, $\frac{\partial H}{\partial x} > 0$, $\frac{\partial H}{\partial y} < 0$ and $\frac{\partial H}{\partial z} > 0$. At $t=0$, is $\frac{dH}{dt}$ positive, zero, or negative?

   At $t=0$, $\frac{dH}{dt} = 2\frac{\partial H}{\partial x} + \frac{\partial H}{\partial z} > 0$.

3. Let

$$f(x,y) = x^3 - xy + \cos(\pi(x+y)).$$

(a) Find a vector normal to the level curve $f(x,y) = 1$ at the point where $x = 1$, $y = 1$.

The gradient of f is normal to the level curve at each point. We find $\operatorname{grad} f(x,y) = (3x^2 - y - \pi\sin(\pi(x+y)))\vec{i} + (-x - \pi\sin(\pi(x+y)))\vec{j}$, and $\operatorname{grad} f(1,1) = 2\vec{i} - \vec{j}$.

(b) Find the equation of the line tangent to the level curve $f(x,y) = 1$ at the point where $x = 1$, $y = 1$.

The line is

$$2(x-1) - (y-1) = 0, \quad \text{or} \quad 2x - y = 1.$$

(c) Find a vector normal to the graph $z = f(x,y)$ at the point $x = 1$, $y = 1$.

The graph is the level surface $g(x,y,z) = 0$ of the function $g(x,y,z) = f(x,y) - z$. The gradient of g is normal to the level surface at each point. We have $\operatorname{grad} g(x,y,z) = \operatorname{grad} f(x,y) - \vec{k}$. Now $f(1,1) = 1$, so a vector normal to the graph at $(1,1,1)$ is

$$\operatorname{grad} g(1,1,1) = \operatorname{grad} f(1,1) - \vec{k} = 2\vec{i} - \vec{j} - \vec{k}.$$

(d) Find the equation of the plane tangent to the graph $z = f(x,y)$ at the point $x = 1$, $y = 1$.

The plane is $2(x-1) - (y-1) - (z-1) = 0$, or $2x - y - z = 0$.

4. Let

$$f(x,y) = (x-y)^3 + 2xy + x^2 - y.$$

(a) Find the linear approximation $L(x,y)$ near the point $(1,2)$.

First get the numbers: $f(1,2) = -1 + 4 + 1 - 2 = 2$,
$f_x(x,y) = 3(x-y)^2 + 2y + 2x$, $f_x(1,2) = 3 + 4 + 2 = 9$,
$f_y(x,y) = -3(x-y)^2 + 2x - 1$, $f_y(1,2) = -3 + 2 - 1 = -2$.
Then $L(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = 2 + 9(x-1) - 2(y-2)$.

(b) Find the quadratic approximation $Q(x,y)$ near the point $(1,2)$.

We need some more numbers:
$f_{xx}(x,y) = 6(x-y) + 2$, $f_{xx}(1,2) = -6 + 2 = -4$,
$f_{xy}(x,y) = -6(x-y) + 2$, $f_{xy}(1,2) = 6 + 2 = 8$,
$f_{yy}(x,y) = 6(x-y)$, $f_{xy}(1,2) = -6$.
Then

$$\begin{aligned} Q(x,y) &= L(x,y) + \frac{f_{xx}(1,2)}{2}(x-1)^2 + f_{xy}(1,2)(x-1)(y-2) + \frac{f_{yy}(1,2)}{2}(y-2)^2 \\ &= 2 + 9(x-1) - 2(y-2) - 2(x-1)^2 + 8(x-1)(y-2) - 3(y-2)^2. \end{aligned}$$

5. For each of the following functions, determine the set of points where the function is *not* differentiable. *Briefly* explain how you know it is not differentiable; use a picture if it helps.

   (You do not have to *prove* that it is not differentiable; just identify the set of points based on your understanding of what differentiable means.)

   (a) $f(x, y) = \left|x^2 + y^2 - 1\right|$

This function is not differentiable on the circle $x^2 + y^2 = 1$. The graph has a "corner" at these points.

   (b) $f(x, y) = (x^2 + y^2)^{1/4}$

This function is not differentiable at the origin. Consider the cross section $y = 0$: $f(x, 0) = (x^2)^{1/4} = \sqrt{\lvert x \rvert}$. The graph has a cusp (i.e. a point) at $x = 0$.

   (c) $f(x, y) = e^{-x^2+y}$

This function is the composition of polynomials and the exponential function, so it is differentiable everywhere.

   (d) $f(x, y) = \dfrac{x^3 - xy + 1}{x^2 - y^2}$

This function is not differentiable at points where the denominator is zero; that is, where $x^2 = y^2$. This gives the lines $y = x$ and $y = -x$.

6. An assortment of TRUE/FALSE or short answer questions:

   (a) TRUE or FALSE: If f is differentiable at $(0, 0)$, then f is continuous at $(0, 0)$.

TRUE. See the middle box on page 692.

   (b) TRUE or FALSE: If f is a continuous function defined on the region $x^2 + y^2 \leq 9$, then f has a maximum value and a minimum value in this region.

TRUE. Since f is continuous, and the region is closed and bounded, Theorem 15.1 (the Extreme Value Theorem, page 716) shows that f has a global maximum and a global minimum in the region.

   (c) TRUE or FALSE: If $f_x(0, 0)$ exists, and $f_y(0, 0)$ exists, then f is differentiable at $(0, 0)$.

FALSE. $f(x, y) = x^{1/3}y^{1/3}$ is a counter-example (see Example 3, page 691, or your notes).

   (d) TRUE or FALSE: If f is differentiable at $(0, 0)$, then the tangent plane to the graph of f at $(0, 0)$ is given by $z = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y$.

TRUE. See Example 1, page 690, and the comment just above Example 1.

   (e) Give an example of a function $f(x, y)$ for which $(0, 0)$ is a local minimum, but for which the second derivative test fails to determine this classification.

$f(x, y) = x^4 + y^4$ (see Example 7, page 708, or your notes).

   (f) Give an example of a function $g(x, y)$ which is differentiable everywhere except along the line $y = x$.

$g(x, y) = \lvert x - y \rvert$. The graph consists of two planes ($z = x - y$ if $x \geq y$, and $z = y - x$ if $x < y$) that meet in a corner along $y = x$.

(g) Let $H(x,y) = x^2 - y^2 + xy$, and suppose that x and y are both functions that depend on t. Express $\dfrac{dH}{dt}$ in terms of x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} = (2x+y)\frac{dx}{dt} + (-2y+x)\frac{dy}{dt}$

7. Suppose f is a differentiable function such that

$$f(1,3) = 1, \quad f_x(1,3) = 2, \quad f_y(1,3) = 4,$$

$$f_{xx}(1,3) = 2, \quad f_{xy}(1,3) = -1, \quad \text{and} \quad f_{yy}(1,3) = 4.$$

(a) Find $\text{grad} f(1,3)$.

$\text{grad} f(1,3) = f_x(1,3)\vec{i} + f_y(1,3)\vec{j} = 2\vec{i} + 4\vec{j}$

(b) Find a vector in the plane that is perpendicular to the contour line $f(x,y) = 1$ at the point $(1,3)$.

$2\vec{i} + 4\vec{i}$ (from (a); the gradient vector at a point is perpendicular to the contour line through that point)

(c) Find a vector that is perpendicular to the surface $z = f(x,y)$ (i.e. the graph of f) at the point $(1,3,1)$.

The graph is the level surface $g(x,y,z) = 0$ of the function $g(x,y,z) = f(x,y) - z$. The gradient of g is normal to the level surface at each point. We have $\text{grad}\, g(x,y,z) = \text{grad}\, f(x,y) - \vec{k}$. Now $f(1,3) = 1$, so a vector normal to the graph at $(1,3,1)$ is

$$\text{grad}\, g(1,3,1) = \text{grad}\, f(1,3) - \vec{k} = 2\vec{i} + 4\vec{j} - \vec{k}.$$

(d) At the point $(1,3)$, what is the rate of change of f in the direction $\vec{i} + \vec{j}$?

$\vec{u} = (\vec{i} + \vec{j})/\sqrt{2}$ is a unit vector in the direction of $\vec{i} + \vec{j}$. The rate of change of f in this direction is $f_{\vec{u}}(1,3) = \text{grad} f(1,3) \cdot \vec{u} = (2\vec{i} + 4\vec{j}) \cdot (\vec{i} + \vec{j})/\sqrt{2} = 6/\sqrt{2} = 3\sqrt{2}$.

(e) Use a quadratic approximation to estimate $f(1.2, 3.3)$.

Near $(1,3)$, we have

$$f(x,y) \approx f(1,3) + f_x(1,3)(x-1) + f_y(1,3)(y-3) + \frac{f_{xx}(1,3)}{2}(x-1)^2 + f_{xy}(1,3)(x-1)(y-3) + \frac{f_{yy}(1,3)}{2}(y-3)^2.$$

So $f(1.2, 3.3) \approx 1 + (2)(0.2) + (4)(0.3) + (2/2)(0.2)^2 + (-1)(0.2)(0.3) + (4/2)(0.3)^2 = 2.76$.

8. For each of the following functions, find and classify the critical points.

(a) $f(x,y) = x^3 - x^2 + 2xy + 2y^2$

$f_x(x,y) = 3x^2 - 2x + 2y$ and $f_y(x,y) = 2x + 4y$. We must solve (i) $3x^2 - 2x + 2y = 0$ and (ii) $2x + 4y = 0$. (ii) gives us $y = -x/2$, and substituting this into (i) gives $3x^2 - 3x = 0$. This gives $x = 0$ or $x = 1$. If $x = 0$ then $y = 0$, and if $x = 1$ then $y = -1/2$. So the critical points are $(0,0)$ and $(1,-1/2)$.

For the second derivative test, we need $f_{xx}(x,y) = 6x - 2$, $f_{xy}(x,y) = 2$, and $f_{yy}(x,y) = 4$.

At $(0,0)$, $D = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = (-2)(4) - 2^2 = -12 < 0$, so $(0,0)$ is a saddle point.

At $(1,-1/2)$, $D = f_{xx}(1,-1/2)f_{yy}(1,-1/2) - f_{xy}(1,-1/2)^2 = (4)(4) - 2^2 = 12 > 0$, and $f_{xx}(1,-1/2) = 4 > 0$, so $(1,-1/2)$ is a local minimum.

(b) $g(x,y) = \sqrt{(x-1)^2 + y^2}$

$g_x(x,y) = \frac{x-1}{\sqrt{(x-1)^2+y^2}}$, and $g_y(x,y) = \frac{y}{\sqrt{(x-1)^2+y^2}}$. Note that the denominators are zero when $x = 1$ and $y = 0$. Therefore g_x and g_y are not defined there, so $(1,0)$ is a critical point. Solving $g_x = 0$ and $g_y = 0$ does not give any more points, so the only critical point is $(1,0)$.

This function is not differentiable at $(1,0)$, so we can not use the second derivative test. However, we recognize the shape of the graph: it is a cone, and the point of the cone is at $(1,0)$. (See, for example, Example 4 on page 578; the cone in this problem has been shifted to $(1,0)$.) Therefore $(1,0)$ is a local (and global) minimum.

(c) $h(x,y) = e^{-x+y} + x + y^2$

$h_x(x,y) = -e^{-x+y} + 1$, and $h_y(x,y) = e^{-x+y} + 2y$. To find the critical points, we must solve (i) $-e^{-x+y} + 1 = 0$ and (ii) $e^{-x+y} + 2y = 0$. First (i) gives us $e^{-x+y} = 1$, which implies $-x + y = 0$, so $y = x$. Substitute this into (ii) to get $1 + 2y = 0$, and therefore $y = -1/2$. So the only critical point is $(-1/2,-1/2)$.

We need $h_{xx}(x,y) = e^{-x+y}$, $h_{xy}(x,y) = -e^{-x+y}$, and $h_{yy}(x,y) = e^{-x+y} + 2$.

At $(-1/2,-1/2)$, we have $D = h_{xx}(-1/2,-1/2)h_{yy}(-1/2,-1/2) - h_{xy}(-1/2,-1/2)^2 = (1)(3) - (-1)^2 = 2 > 0$, and $h_{xx}(-1/2,-1/2) = 1 > 0$, so $(-1/2,-1/2)$ is a local minimum.

(d) $r(x,y) = (x-y)(x+y)x$ (Hint: Consider the contour lines $r(x,y) = 0$.)

Let's ignore the hint for the moment. Let's rewrite $r(x,y) = x^3 - xy^2$. We have $r_x(x,y) = 3x^2 - y^2$ and $r_y(x,y) = -2xy$. We must solve (i) $3x^2 - y^2 = 0$ and (ii) $-2xy = 0$. But (ii) implies that either $x = 0$ or $y = 0$. If $x = 0$, then (i) implies $y = 0$. On the other hand, if $y = 0$, then (i) implies $x = 0$. Thus the only critical point is $(0,0)$.

We need $r_{xx}(x,y) = 6x$, $r_{xy}(x,y) = -2y$, and $r_{yy}(x,y) = -2x$.

At $(0,0)$, we have $D = r_{xx}(0,0)r_{yy}(0,0) - r_{xy}(0,0)^2 = 0$. So the second derivative test does not tell us the shape of the graph near $(0,0)$.

Now let's use the hint. Consider the contour lines given by $r(x,y) = 0$. This is the set of points where $(x-y)(x+y)x = 0$. This is three lines through the point $(0,0)$: $y = x$, $y = -x$, and $x = 0$. Since these are the lines where $r = 0$, they are the boundaries of the regions where $r > 0$ and $r < 0$. Since the three lines all go through $(0,0)$, they separate the plane into six wedge-shaped regions, and the sign of r alternates from region to region (i.e. there are three regions where $r > 0$ and three regions where $r < 0$). In other words, the shape of the graph is a monkey saddle.