Math 214 Linear Algebra **Quiz 3 Solutions** **Spring 2005**

1. Let V be the vector space of functions spanned by $S = \{\cos(t), \sin(t), \cos(2t), \sin(2t)\}$.
   Let $L : V \to V$ be defined as $L(f(t)) = f'(t)$. (L is the differentiation operator.)

   (a) Find the matrix representation of L with respect to S.
   Solution:

$$L(\cos(t)) = -\sin(t),$$
$$L(\sin(t)) = \cos(t),$$
$$L(\cos(2t)) = -2\sin(2t),$$
$$L(\sin(2t)) = 2\cos(2t),$$

   and by inspection,

$$[L(\cos(t))]_S = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad [L(\sin(t))]_S = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$[L(\cos(2t))]_S = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \quad [L(\sin(2t))]_S = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

   The matrix representation of L with respect to S is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

   (b) Is L an isomorphism?
   Solution:
   Yes. L is linear; we must show that L is one-to-one and onto. We can do this by showing that A is nonsingular. We can do this by observing that A is row-equivalent to the identity matrix, or by actually finding the inverse, which is

$$A^{-1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}$$

2. Find the area of the triangle in the plane formed by the points $(1,1)$, $(3,2)$ and $(7,6)$.

   Solution:

   There are (at least) two ways to answer this question. Here is one:

   The displacement vector from $(1,1)$ to $(3,2)$ is $\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The displacement vector from $(1,1)$ to $(7,6)$ is $\vec{\mathbf{w}} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$. Put these vectors in a matrix, and take the determinant to find the area of the parallelogram formed by the vectors:

   $$A = \begin{bmatrix} 2 & 6 \\ 1 & 5 \end{bmatrix}, \quad \det(A) = 4.$$

   The triangle is half of the parallelogram, so the area of the triangle is $\boxed{\mathbf{2}}$.

   (Another method is given on pages 376–378 of the text.)

3. Find the values of λ where the following matrix is singular.

   $$A = \begin{bmatrix} \lambda - 1 & 2 & 3 \\ 0 & \lambda & 0 \\ -1 & -3 & \lambda - 5 \end{bmatrix}$$

   Solution:

   The matrix is singular if the determinant is zero. By using a cofactor expansion down the first column, we find

   $$\begin{aligned} \det(A) &= (\lambda - 1)(\lambda(\lambda - 5)) - (-3\lambda) \\ &= \lambda(\lambda^2 - 6\lambda + 8) \\ &= \lambda(\lambda - 2)(\lambda - 4) \end{aligned}$$

   This is zero (and therefore the matrix is singular) when $\boxed{\lambda = 0, \lambda = 2 \text{ or } \lambda = 4}$.

4. This problem contains assorted short answer or true/false questions.

   In all cases, A and B are $n \times n$ matrices.

   Briefly explain or justify your answers.

   (a) Suppose $\det(A) = r$. Find $\det(cA)$, where c is a number.

   Solution:

   If we multiplied *one* column (or row) of A by c, the determinant would change by the factor c. In this case, we are multiplying each column (or row) by c, and there are n columns, so

$$\det(cA) = c^n \det(A) = c^n r$$

   OR:

$$\det(cA) = \det(cIA) = \det(cI)\det(A),$$

   and $\det(cI) = c^n$, so again we find

$$\det(cA) = c^n r.$$

   (b) Suppose A is a nonsingular matrix. Show that $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

   Solution:

   We have

$$\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1,$$

   therefore

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

   (c) True or False: $\det(A+B) = \det(A) + \det(B)$.

   False.

   Here is a counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\det(A) = 0$ and $\det(B) = 0$, but $\det(A+B) = \det(I) = 1$.

   (d) True or False: $\det(A^2) = (\det(A))^2$.

   True.

$$\det(A^2) = \det(AA) = \det(A)\det(A) = (\det(A))^2.$$

   (e) True or False: If A is row equivalent to B, then $\det(A) = \det(B)$.

   False.

   Here is a counterexample. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and $B = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. A is row equivalent to I, but $\det(A) = 4$ and $\det(B) = 1$.