Math 307 - Dynamical Systems and Chaos **Fall 2005**

Homework 3

Due *Friday, October 21*

Some of the following definitions are new; some are slight variations of a definitions we've already seen; and some are just convenient reminders.

Definition. In a metric space X with metric d, the triangle inequality says

$$d(x, z) \leq d(x, y) + d(y, z)$$

for $x, y, z \in X$.

Definition. Let f be a mapping that is not necessarily invertible. We define $f^{-1}(p)$ to be all the points $y \in X$ for which $f(y) = p$. Similarly, if U is a subset of X, we define $f^{-1}(U)$ to be all the points $y \in X$ for which $f(y) \in U$. For example, if $f(x) = x^2$, then $f^{-1}(4) = \{-2, 2\}$. Also, if U is the open interval $(1, 9)$, then $f^{-1}(U) = (-3, -1) \cup (1, 3)$.

Definition. Let $f : X \to Y$. f is continuous at p if, for any sequence $\{p_i\}_{i=0}^{\infty}$ with $\lim_{i\to\infty} p_i = \overline{p}$, we have $\lim_{i\to\infty} f(p_i) = f(\overline{p})$.

Definition. Recall that for a point p in the real line, $N_\delta(p) = \{x \in J \mid p - \delta < x < p + \delta\}$. In the more general setting of a metric space X with metric d,

$$N_\delta(p) = \{x \in X \mid d(p, x) < \delta\}.$$

Note that $N_\delta(p)$ is an open set.

Definition. The distance between two *sets* is defined to be minimum of the distances between any point in the first set and any point in the second.

---

1. Suppose $f : X \to Y$ is a continuous function from X to Y.

   Let U be an open subset of Y. Show that $f^{-1}(U)$ is an open subset of X. (In words, *the inverse images of open sets are open.*)

2. Show that, for any $w, x, y, z \in X$,

$$d(w, z) \leq d(w, x) + d(x, y) + d(y, z) \quad (1)$$

   which implies

$$d(w, x) \geq d(w, z) - d(x, y) - d(y, z). \quad (2)$$

   Make a sketch that illustrates the inequality (1). (The text focuses on subsets of the real line, but a diagram in the plane will probably be clearer.) Hint: This is just a variation of the triangle inequality.

3. Suppose that $d(x,z) > 2\delta$ for some constant $\delta > 0$. Show that for any $y \in X$, either $d(x,y) > \delta$ or $d(z,y) > \delta$. (That is not an *exclusive* or; both $d(x,y)$ and $d(z,y)$ could be greater than δ. The question asks you to show that at least one of them *must* be greater than δ.)

4. Let $q \in X$, and let $n > 0$ be some integer. Let

$$V = \bigcap_{i=0}^{n} f^{-i}\left(N_\delta\left(f^i(q)\right)\right). \tag{3}$$

   Describe the set V. What do the points in this set have in common?

   Hint: The first term in the intersection ($i = 0$) is simply $N_\delta(q)$. Now, $N_\delta(f(q))$ is a neighborhood of $f(q)$, and the second term in the intersection is $f^{-1}(N_\delta(f(q)))$. What does the intersection of this term and $N_\delta(q)$ give? Generalize to the intersection of the $n+1$ terms.

   This problem is just asking you to carefully parse and explain the meaning of the set defined in equation (3).

5. To prove that a map $f : X \to X$ has sensitive dependence on initial conditions, we must show that there is a $\delta > 0$ such that, for every $x \in X$, every neighborhood W of x contains a point z such that

$$d(f^m(x), f^m(z)) > \delta \tag{4}$$

   for some integer $m \geq 0$. (I've used the distance function d here; on the real line, this would be $|f^m(x) - f^m(z)| > \delta$.)

   Suppose that the continuous map $f : X \to X$ is topologically transitive and has a dense set of periodic orbits. In this problem, you will show that f must also have sensitive dependence on initial conditions. To do this, you will develop the proof in a sequence of steps. (Since the problem is a bit wordy, I've highlighted in blue the parts that you must complete.)

   (a) Show that there is a number ρ such that for all $x \in X$, there exists a periodic point $q \in X$ whose orbit $O^+(q)$ is of distance at least $\rho/2$ from x. (In other words, the distance from x to each point in the orbit of q is at least $\rho/2$.)

   Hint: Choose two arbitrary periodic points q_1 and q_2 with disjoint orbits $O^+(q_1)$ and $O^+(q_2)$. Let ρ be the distance between $O^+(q_1)$ and $O^+(q_2)$.

   (b) Let $x \in X$ be an arbitrary point in X, let W be some neighborhood of x. Let $\delta = \rho/8$, where ρ is the number from part (a). Let $U = W \cap N_\delta(x)$.

   Briefly explain why U must contain a periodic point p.

   (c) Let n be the period of the periodic point p of part (b). Let q be a periodic point whose orbit is of distance at least 4δ away from x. (See the figure below. That such an orbit exists was shown in part (a), since $4\delta = \rho/2$.) Let V be the set defined in equation (3) of question 4. V is open (by the result of question 1), and V is non-empty (it contains q). Because f is topologically transitive, there exists $y \in U$ and positive integer k such that $f^k(y) \in V$.

   (At this point, we have a period-n point $p \in U$, and a point $y \in U$ whose orbit enters V in k iterations. See the figure below for an illustration.)

During the iterations from $i = 0$ to $i = k$, $f^i(y)$ is on its way to V, while $f^i(p)$ is iterating around a periodic orbit. After each n iterations, this periodic orbit returns to p:

$$f^{nj}(p) = p$$

for any integer $j \geq 0$. Let j be the first integer such that $nj \geq k$.
Show that $f^{nj}(y) \in N_\delta\left(f^{nj-k}(q)\right)$.

(d) Show that $d(p, f^{nj}(y)) > 2\delta$. Hint: The figure below might help you "see" what is going on. To write the proof, you might use the result in question 2, applied to the points p, $f^{nj}(y)$, $f^{nj-k}(q)$, and x.

(e) Finish the proof by applying the result of question 3 to the previous step. Which point(s) correspond to z in the equation (4), and what is the value of m? Hint: Recall that $f^{nj}(p) = p$, so the previous step showed that $d(f^{nj}(p), f^{nj}(y)) > 2\delta$. Use the triangle inequality on p, $f^{nj}(y)$, and $f^{nj}(x)$.

The figure below is an attempt to illustrate some of the definitions in the proof.