**4.12(a)** Inspired by the hint in the back of the book, we'll try $\varepsilon_n = \frac{1}{n(n+1)}$. Note that

$$\varepsilon_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

so $\varepsilon_n$ is the distance between $1/(n+1)$ and $1/n$. With intervals of length $\varepsilon_n$, we can cover the points in the interval $[0, 1/(n(n+1))]$ with one interval. This leaves $n(n+1) - 1$ points left to be covered. Because $\varepsilon_n$ is the distance between $1/(n+1)$ and $1/n$, the $n$ points $1, 1/2, 1/3, \ldots, 1/n$ require $n$ intervals to cover them. The remaining points lie between $\frac{1}{n(n+1)}$ and $\frac{1}{n+1}$. The distance between these points is

$$\frac{1}{n+1} - \frac{1}{n(n+1)} = \frac{n-1}{n(n+1)} = (n-1)\varepsilon_n,$$

so we can cover these with $n-1$ intervals. Adding up all the intervals gives $N(\varepsilon_n) = 1+n+(n-1) = 2n$. Then we have

$$\text{boxdim}(S) = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(\varepsilon^{-1})} = \lim_{n\to\infty} \frac{\ln(2n)}{\ln(n(n+1))} = \lim_{n\to\infty} \frac{\left(\frac{2}{2n}\right)}{\left(\frac{2n+1}{n(n+1)}\right)} = \lim_{n\to\infty} \frac{n^2+n}{2n^2+n} = \frac{1}{2},$$

where L'Hospital's rule has been used in the last three equalities.

**4.12(b)** Let $\varepsilon_n = (1/2)^n$. We can cover the poins in $[0, \varepsilon_n]$ with one interval. There are $n$ points left to be covered, and they are all separated by distances greater than $\varepsilon_n$, so we need $n$ intervals to cover these. Therefore, $n+1$ intervals are required to cover all the points, and

$$\text{boxdim}(S) = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(\varepsilon^{-1})} = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln(2^n)} = \lim_{n\to\infty} \frac{\ln(n+1)}{n \ln 2} = \lim_{n\to\infty} \frac{1}{(\ln 2)(n+1)} = 0.$$

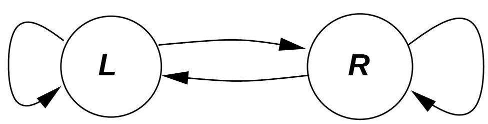
**5.5** We know that each bi-infinite itinerary corresponds to a single point in the invariant set of the horseshoe map. Consider just the right side of the bi-infinite sequences. These infinite sequences can be put into a one-to-one correspondence with the numbers in the interval $[0, 1]$, by identifying the symbols $L$ and $R$ with the digits 0 and 1 in the binary representation of a number in $[0, 1]$. The eventually periodic itineraries correspond to rational numbers, and the orbits that are not eventually periodic (which are the itineraries of chaotic orbits) correspond to the irrational numbers. There are uncountably many irrational numbers in $[0, 1]$, so there are uncountably many chaotic orbits.

**5.6** Construct an infinite symbol sequence of $L$ and $R$ by concatenating all the possible sequences of length 2, then all the possible sequences of length 4, then all of length 6, and so on. Make this the right half of a bi-infinite itinerary, so the right half might be

$$.LL\ LR\ RL\ RR\ LLLL\ LLLR\ LLRL\ LLRR\ LRLL\ LRLR\ LRRL\ LRRR\ RLLL\ldots$$

(The spaces are there to make the finite sequences readable.) Make the left half of the bi-infinite itinerary anything you like. This orbit will visit the rectangular region labeled $L.L$, then in two iterations $L.R$, then in two more $R.L$, and so on. (See Figure 5.18(a) for a picture of $L.L$, etc.) To see that the orbit is dense in the invariant set, let $(x_0, y_0)$ be a point in the invariant set, and let $\varepsilon > 0$ be a small number. By considering left and right finite sequences $S_{-n+1} \ldots S_0.S_1S_2 \ldots S_n$ of sufficient length, we can find a rectangle with such a label that is contained within an $\varepsilon$-neighborhood of $(x_0, y_0)$. Since the orbit will eventually land in this rectangle, it will be within $\varepsilon$ of $(x_0, y_0)$. This is true for arbitrary $\varepsilon$ and for any point $(x_0, y_0)$ in the invariant set, so the orbit will come arbitrarily close to every point in the set. Hence the orbit is dense.

**5.8(a)** This map will have the same dynamics as the horseshoe map, but the $y$ coordinate is never inverted. The transition graph is the same as the transition graph of the horseshoe map. As with the horseshoe map, we form the Markov paritition by dividing the square into two pieces along a vertical line, and label the left and right halves $L$ and $R$, respectively. $f(L)$ lies across $L$ and $R$, and $f(R)$ lies across $L$ and $R$, so the transition graph is




Drawings of the strips that remains in the square for two forward or backward iterates are shown in the following pages. [1] shows the sets $.R$ and $.L$, which are the points that remain in the square after one iteration. As usual, $.R$ contains the points that will be in $R$ after one iteration, and $.L$ contains the points that will be in $L$. [2] shows the subsets that will remain in the square after two iterations. The drawing isn't perfect; each strip in [2] is a subset of one of the strips in [1]. Continuing this process leads to a Cantor set of horizontal lines.

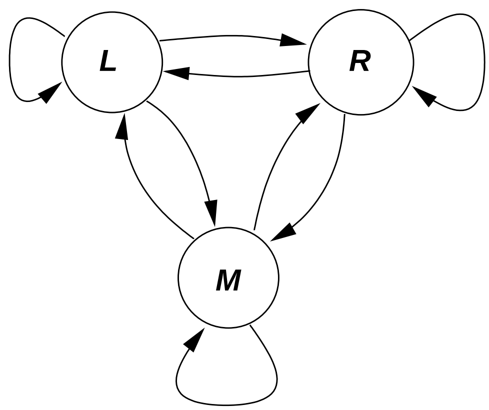
The box labeled [3] shows the Markov partition of the square into $L$ and $R$. [4] shows the sets that remain in the square after one backward interation of the map, and [5] shows those that remain in the square after two backward iterations. Continuing this process leads to a Cantor set of vertical lines.

Any periodic itinerary corresponds to a periodic orbit. For example, the itinerary

$$\ldots LLRRLL.RRLLRR\ldots$$

corresponds to a period 4 orbit. The infinte forward and backward itineraries determine the coordinates of four points that make up the orbit. The $x$ coordinates are given by $x_1 = .RRLLRR\ldots$, $x_2 = .RLLRRL\ldots$, $x_3 = .LLRRLL\ldots$, and $x_4 = .LRRLLR\ldots$. Similarly, the $y$ coordinates are $y_1 = \ldots LLRRLL.$, $y_2 = \ldots LRRLLR.$, $y_3 = \ldots RRLLRR.$ and $y_4 = \ldots RLLRRL.$. We can roughly desribe the location and pattern of this orbit by considering the subsets labeled with just two symbols (forward or backward). If we overlay $\boxed{4}$ on top of $\boxed{2}$, we obtain 16 small rectangles, labeled $LL.LL$, $RL.LL$, etc. (This is a picture like Figure 5.18(a), but with finer resolution and with some of the boxes label differently.) The period four orbits visits $LL.RR$, $LR.RL$, $RR.LL$ and $RL.RL$, in that order. Of course, by considering longer symbols sequences, we can locate the points in the orbit more precisely.

**5.8(b)** We can analyze this map in the same way that we analyzed the horseshoe map, but we will have *three* symbols instead of two. We form the Markov partition by splitting the square into vertical strips, labeled $L$, $M$ and $R$. Since each subset in the partition lies across all the other subsets (i.e. $f(L)$ lies across $L$, $M$ and $R$, as does $f(M)$ and $f(R)$), we have the following transition graph:




The symbolic itineraries will consist of arbitrary bi-infinite sequences of the three symbols.

Drawings of the sets that remain in the square for two forward or backward iterations are shown on a following page. $\boxed{1}$ shows the sets labels $.L$, $.M$ and $.R$. These are the points that remain in the square after one iteration. $\boxed{2}$ shows the sets that remain after two iterations. There are 9 horizontal strips. As usual, continuing this process leads to a Cantor set of horizontal lines. Each horizontal line corresponds to an infinte sequence of the symbols $L$, $M$ and $R$.

$\boxed{3}$ shows the Markov partition, and $\boxed{4}$ shows the sets of points that remain in the square after one backward iteration. The set that remains after two backward iterations consists of 27 thin vertical strips; each of these is a subset of one of the strips in $\boxed{3}$. For example, the strip labeled $RM.$ in $\boxed{4}$ contains the strips labeled $RRM.$, $MRM.$, and $LRM.$ (left to right), while the strip labeled $MM.$ contains the strips labels $LMM.$, $MMM.$, $RMM.$ (left to right). In the limit we obtain a Cantor set of vertical lines, one for each infinite sequence of symbols.

As for the horseshoe map and the map in part (a), any periodic bi-infinite sequence of symbols corresponds to a periodic orbit of this map. For example, the itinerary

$$\ldots LLMRLLMR.LLMRLLMR\ldots$$

corresponds to a period 4 orbit.