Math 307: Dynamical Systems and Chaos
Problem Set 6 Solutions

T6.1 The statement is not necessarily true if f is not continuous, so we assume that f is continuous. The proof of this statement becomes simple if we make two observation, one about the definition of the forward limit set, and another about continuous functions.

From the definition of the forward limit set, it follows that $x_0 \in \omega(y)$ if and only if the orbit $\{f^n(y) : n \geq 0\}$[1] contains a subsequence, say $\{z_n\}$, that converges to x_0. To see that this follows from the definition in the text, let $\varepsilon_n = 2^{-n}$ (or any other sequence of numbers that converges to zero), and for each ε_n, let z_n be a point such that $|f^n(y) - z_n| < \varepsilon_n$. By the definition of $\omega(y)$, such a number must exist. (In some texts, this is how the forward limit set is *defined*: a point x_0 is in $\omega(y)$ if the orbit of y contains a subsequence that converges to x_0.)

A property of continuous functions is that if $f(x)$ is continuous, and if the sequence $\{x_n\}$ converges to p, then the sequence $\{f(x_n)\}$ converges to $f(p)$.

Now we'll use these facts to prove the statement. Suppose $x_0 \in \omega(y)$ for some y. We want to show that the entire orbit of x_0 is in $\omega(y)$. It will be sufficient to show that $f(x_0)$ is in $\omega(y)$, because then we can apply the same argument to $f(x_0)$ to show that $f^2(x_0)$ is in $\omega(y)$, and so on. Since $x_0 \in \omega(y)$, there is a subsequence of the orbit of y that converges to x_0. Call this subsequence $\{z_n\}$. Now consider the sequence $\{f(z_n)\}$. This is also a subsequence of the orbit of y, since each z_n is in the orbit of y. By the continuity of f, the sequence $\{f(z_n)\}$ converges to $f(x_0)$. Since $\{f(z_n)\}$ is a subsequence of the orbit of y, $f(x_0)$ is in the forward limit set of y.

T6.2 Suppose x_0 is a fixed point sink. By the definition of a sink, there is a neighborhood U of x_0 such that if $x \in U$, $f^n(x) \to x_0$ as $n \to \infty$. Suppose $x_0 \in \omega(y)$ for some y. By the definition of the forward limit set, there exists a point in the orbit of y that must be in U. This implies that the orbit converges to x_0, so no other point can also be in the forward limit set of y.

If that last sentence is not convincing, here is a more detailed argument. Consider any point $x_1 \neq x_0$. Let $\varepsilon = |x_1 - x_0|/2$. Because the orbit of y converges to x_0, there is an N such that for all $n > N$, $|f^n(y) - x_0| < \varepsilon$. This implies that for all $n > N$, $|f^n(y) - x_1| > \varepsilon$, so x_1 can not be in the forward limit set of y.

Can a forward limit set contain any fixed points and be more than a single point?

Yes. Consider the tent map

$$T_2(x) = \begin{cases} 2x & x \leq 1/2 \\ 2(1-x) & x > 1/2 \end{cases}$$

(See Example 3.8 for a review of T_2.) This map has chaotic orbits, and it has orbits that are dense in the unit interval $[0, 1]$. Therefore the interval $[0, 1]$ is a forward limit set. This set contains the fixed point $x = 2/3$.

Another example is the interval $[0, 1]$ for the logistic map $G(x) = 4x(1 - x)$. This interval is a forward limit set of a dense orbit, and it contains the fixed point $x = 3/4$.

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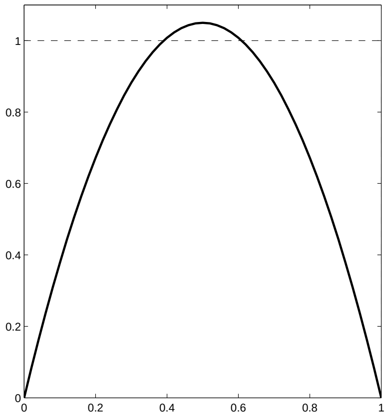
[1] From here on, I won't include "$n \geq 0$", and will simply use brackets to indicate sets in which n varies from 0 to infinity.

T6.3(a) When $a > 4$, the graph of g_a looks like this:



Let u_1 and u_2 be the points where $g_a(x) = 1$. If $x \in (u_1, u_2)$, then $f(x) > 1$, and the orbit of x will eventually diverge to $-\infty$.

We'll try to apply the usual arguments. Let $L = [0, 1/2]$ and $R = [1/2, 1]$. Then $f(L) \supseteq L$, $f(L) \supseteq R$, $f(R) \supseteq L$, and $f(R) \supseteq R$. Thus the transition diagram is the same as that for the case $a = 4$; see Figure 1.13 in the text. This implies that there are periodic orbits of all periods. So while the orbits of many (in fact, most) points in $[0, 1]$ will leave $[0, 1]$, there is an invariant set that contains infinitely many periodic orbits.

Next we want to show that the invariant set contains more than just periodic orbits. In fact, we want to show that it contains chaotic orbits. To do this, we want to show that each infinite symbolic itinerary corresponds to a single point in $[0, 1]$. As usual, we start by labeling intervals with finite sequences of symbols (as in Figure 1.12, for example). The hard part here is to show that as the length of the symbols sequence increases, the length of the corresponding interval goes to zero, so that any infinite symbol sequence corresponds to a single point. This is not so easy.

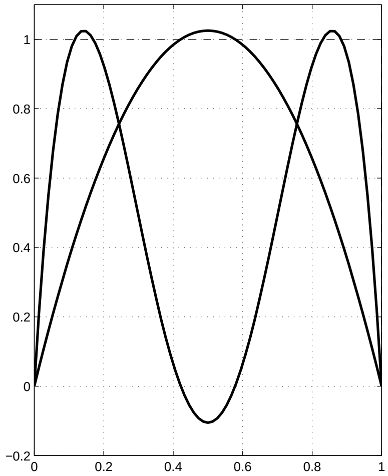
Let's first answer a simpler question. We will show that $g_a(x) = ax(1 - x)$ has a chaotic set for $a > 2 + \sqrt{5} \approx 4.236$. Let r_1 and r_2 be the points where $|g_a'(x)| = 1$. When $a > 2 + \sqrt{5}$, $|g_a'(x)| > 1$ for all x in $[0, r_1] \cup [r2, 1]$. (In other words, we have $0 < u_1 < r_1 < r_2 < u_2 < 1$.) Let α be the minimum value of $g_a'(x)$ on $[0, u_1]$. By our assumption on a, we have $\alpha > 1$. Then any orbit that remains in $[0, r_1] \cup [r_2, 1]$ must have a Lyapunov exponent greater than $\ln(\alpha) > 0$.

Also, because all intervals in $[0, r_1] \cup [r2, 1]$ are stretched by at least a factor of α (i.e. if I is an interval in $[0, r_1] \cup [r2, 1]$, then the length of $g_a(I)$ at least α times the length of I), the lengths of the intervals labeled with finite symbols sequences approach zero as the lengths of the sequences increase, and any infinite sequence of symbols corresponds to a single point. Then any symbol sequence that is not eventually periodic (e.g. $LRLLRLLLRLLLLRLLLLLR\ldots$) corresponds to a point whose orbit is chaotic. To show that the invariant set is a chaotic set, we must show that the invariant set is the forward limit set of some chaotic orbit. To do this, we will show that there is a chaotic orbit that is dense in the invariant set. Construct an itinerary by concatenating all the symbols sequences of length 1, then all the symbol sequences of length 2, and all of length 3, and so on. This itinerary contains every possible finite symbol sequence, so the corresponding orbit will visit every interval labeled with a finite symbols sequence. Therefore, this orbit will come arbitrarily close to every point in the invariant set. So the orbit is dense, and its forward limit set is the invariant set. By construction, this orbit is in the invariant set. So the invariant set is a chaotic set.

Is the invariant set is a chaotic attractor? Any point that is not in the invariant set goes off to

$-\infty$, so the invariant set does not attract any points that are not in the set. The invariant set is a Cantor set, similar to the middle-thirds (or more generally, the "middle-β", where $0 < \beta < 1$) Cantor set that we have seen before. While not all Cantor sets have zero length, all the middle-β Cantor sets do, and so does the invariant set for g_a when $a > 4$. Therefore, the invariant set is not a chaotic attractor.

What if $4 < a < 2 + \sqrt{5}$? Here is one possible approach. Instead of analyzing g_a, we look at g_a^2 (the second iterate of g_a). For example, the following plot shows the graphs of g_a and g_a^2 when $a = 4.1$.



There are four intervals where $0 \leq g_a^2(x) \leq 1$. With a little algebra and calculus, we could find the minimum value of a for which the derivative of g_a^2 is greater than one on these intervals; this value is greater than 4 but less then $2 + \sqrt{5}$. We could then apply the same arguments to g_a^2 that we applied to g_a for $a > 2 + \sqrt{5}$. To get even closer to $a = 4$, apply the same idea to higher iterates of g_a.

T6.3(b) We have already shown that the horseshoe map has an invariant set, and the dynamics on this invariant set can be described with bi-infinite sequences of symbols. To show that this invariant set is a chaotic set, we must show that it is the forward limit set of some orbit in the set. This is the case if the invariant set contains a dense orbit. We use a familiar construction to show that there must be a dense orbit. Consider all the finite itineraries of length $2n$ for $n = 1, 2, \ldots$. For example, $n = 1$ gives LL, LR, RL, and RR. Construct the right side of a bi-infinite itinerary by concatenating all the length 2 sequences, then all the length 4 sequences, then length 6, etc. (Or you can make bigger jumps in the sequence lengths: all length 2, then all length 4, then all length 8, etc.) For example,

$$.LL\ LR\ RL\ RR\ LLLL\ LLLR\ LLRL\ LLRR\ LRLL\ \ldots$$

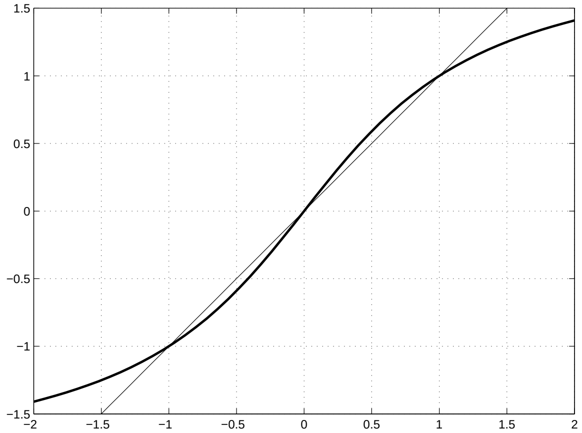
Pick any infinite symbol sequence for the left part of the bi-infinite itinerary. In the first 8 iterations, this orbit is certain to visit the sets $L.L$, $L.R$, $R.L$, and $R.R$. Next it is certain to visit $LL.LL$, $LL.LR$, $LL.RL$, and the rest of the 16 possible sets labeled $S_{-1}S_0.S_1S_2$. Continuing, we see that the orbit corresponding to this bi-infinite itinerary will come arbitrarily close to every point in the invariant set infinitely often, so the orbit is dense. By construction, this orbit is in the invariant set. Therefore the invariant set is a chaotic set.

6.3 This map is really two independent maps, one for x and one for y:

$$x_{n+1} = \frac{4}{\pi}\arctan(x_n), \qquad y_{n+1} = \frac{y}{2}.$$

We can analyze each individually, and then combine the results to understand the two-dimensional map.

The following plot shows the graph of $(4/\pi)\arctan(x)$:



The fixed points are $x = -1$, $x = 0$ and $x = 1$. $x = -1$ and $x = 1$ are sinks, and $x = 0$ is a source. The basin of attraction of -1 is all $x < 0$, and the basin of attraction of $x = 1$ is all $x > 0$.

The y equation is simple: $y = 0$ is a sink, and all orbits converge to $y = 0$ monotonically. (In fact, we have the solution $y_n = y_0/2^n$.)

We now combine these results to describe the behavior of the two-dimensional map $\mathbf{f}$.

- $\mathbf{f}$ has three fixed points: $(-1, 0)$, $(0, 0)$ and $(1, 0)$. These are the only forward limits sets of $\mathbf{f}$.
- The fixed points $(-1, 0)$ and $(1, 0)$ are sinks, so each of them is an attractor.
- The basin of attraction of $(-1, 0)$ is the half plane $x < 0$, and the basin of attraction of $(1, 0)$ is the half plane $x > 0$.
- The fixed point $(0, 0)$ is a saddle point. Its unstable manifold is the set of points on the x axis where $-1 < x < 1$. Its stable manifold is the y axis. The stable manifold of $(0, 0)$ is the basin boundary. All points in the boundary converge to the saddle under iteration of $\mathbf{f}$.