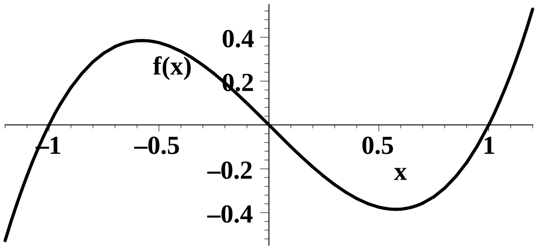
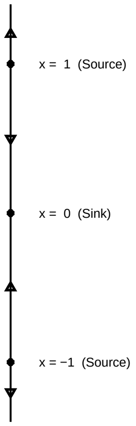
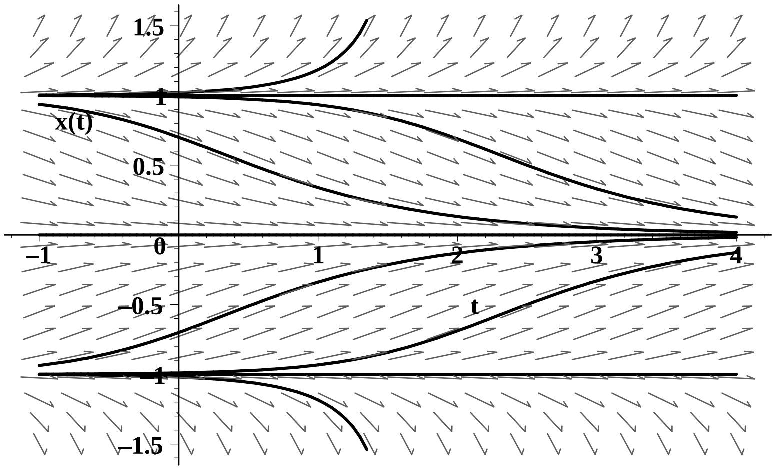
Math 307: Dynamical Systems and Chaos
Problem Set 7 Solutions

T7.1 First, here is the plot of the right hand side of the differential equation.



Next we have the phase portrait (a phase line, in this case), and the slope field. Several solutions are also shown in the slope field, including the equilibrium solutions $x(t) = -1$, $x(t) = 0$ and $x(t) = 1$.





If $-1 \leq x(0) \leq 1$, then $x(t)$ remains bounded, and if $-1 < x(0) < 1$, then $\lim_{t\to\infty} x(t) = 0$.

T7.14 The system is

$$\begin{aligned} \dot{x} &= -x^3 + xy \\ \dot{y} &= -y^3 - x^2 \end{aligned}$$

The origin $(0,0)$ is an equilibrium solution. The Jacobian at $(0,0)$ is

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and the only eigenvalue of this matrix is 0. We can not determine the stability of $(0,0)$ from the linearization.

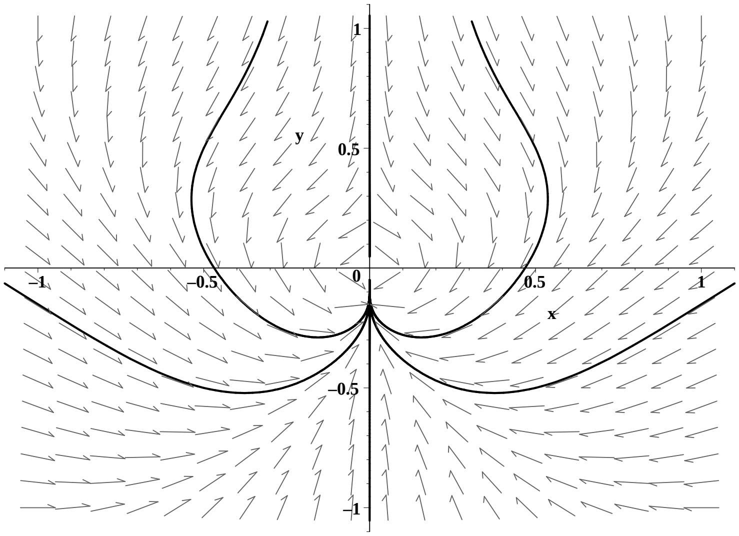
Following the suggestion in the back of the text, we consider the function $E(x,y) = x^2 + y^2$. $E(0,0) = 0$, and $E(x,y) > 0$ for $(x,y) \neq (0,0)$. Note that E is the square of the distance of (x,y) from the origin; if $(0,0)$ is asymptotically stable, it is plausible that this quantity should decrease with time.

Along a solution $(x(t), y(t))$, we find

$$\begin{aligned} \frac{d}{dt}E(x(t),y(t)) &= \frac{\partial E}{\partial x}\dot{x} + \frac{\partial E}{\partial y}\dot{y} \\ &= (2x)(-x^3 + xy) + (2y)(-y^3 - x^2) \\ &= -2x^4 - 2y^4, \\ &= -2(x^4 + y^4) \end{aligned}$$

so $dE/dt < 0$ if $(x,y) \neq (0,0)$. Thus E is a strict Lyapunov function for $(0,0)$. Then, by Theorem 7.23, $(0,0)$ is asymptotically stable.

Here is the phase portrait:



7.3 The system is

$$\begin{aligned} \dot{x} &= 2x - y \\ \dot{y} &= x^2 + 4y \end{aligned}$$

To find the equilibrium solutions, we must solve

$$2x - y = 0 \quad \text{and} \quad x^2 + 4y = 0.$$

The first equation gives $y = 2x$, and substituting this into the second gives $x^2 + 8x = 0$, which has the solutions $x = 0$ or $x = -8$. Therefore there are two equilibria: $(0, 0)$ and $(-8, -16)$.

The Jacobian matrix at an arbitrary point is

$$\begin{pmatrix} 2 & -1 \\ 2x & 4 \end{pmatrix}$$

At $(0, 0)$, we have

$$\begin{pmatrix} 2 & -1 \\ 0 & 4 \end{pmatrix}$$

The eigenvalues of this matrix are 2 and 4, so the linearization at $(0, 0)$ results in a *source.* The equilibrium $(0, 0)$ is *unstable.*

At $(-8, -16)$, we have

$$\begin{pmatrix} 2 & -1 \\ -16 & 4 \end{pmatrix}$$

The eigenvalues of this matrix are $3 + \sqrt{17} \approx 7.123$ and $3 - \sqrt{17} \approx -1.123$, so $(-8, -16)$ is a *saddle point.* It is *unstable.*

7.5(a) The system is

$$\begin{aligned}\dot{x} &= 3x - y \\ \dot{y} &= 2x + 4y\end{aligned}$$

The matrix of this system is

$$A = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix},$$

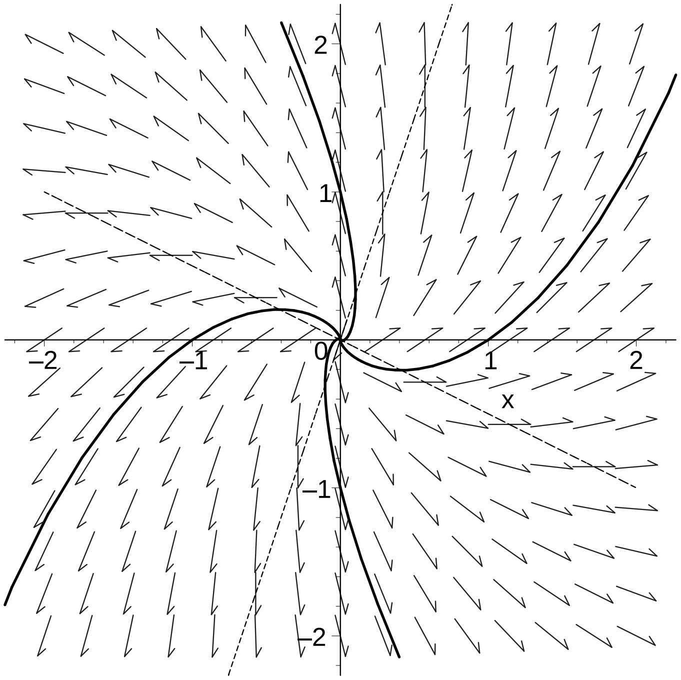
and

$$A - \lambda I = \begin{pmatrix} 3-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix}.$$

The characteristic polynomial is $\det(A - \lambda I) = \lambda^2 - 7\lambda + 14$, and solving $\det(A - \lambda I) = 0$ gives the eigenvalues $\lambda = (7/2) \pm (\sqrt{7}/2)i$. Since the eigenvalues are complex, with positive real part, we know the equilibrium at $(0, 0)$ is a *spiral source.*

The vector field at $(1, 0)$ is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, and at $(0, 1)$ it is $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$. The vector field is vertical along the x-nullcline $3x - y = 0$, and it is horizontal along the y-nullcline $2x + 4y = 0$.

Here is the phase portrait. The dashed lines are the nullclines.



7.5(b) The system is

$$\begin{aligned}\dot{x} &= -2x + 3y\\ \dot{y} &= 7x - 6y\end{aligned}$$

The matrix of this system is

$$A = \begin{pmatrix} -2 & 3 \\ 7 & -6 \end{pmatrix},$$

and

$$A - \lambda I = \begin{pmatrix} -2-\lambda & 3 \\ 7 & -6-\lambda \end{pmatrix}.$$

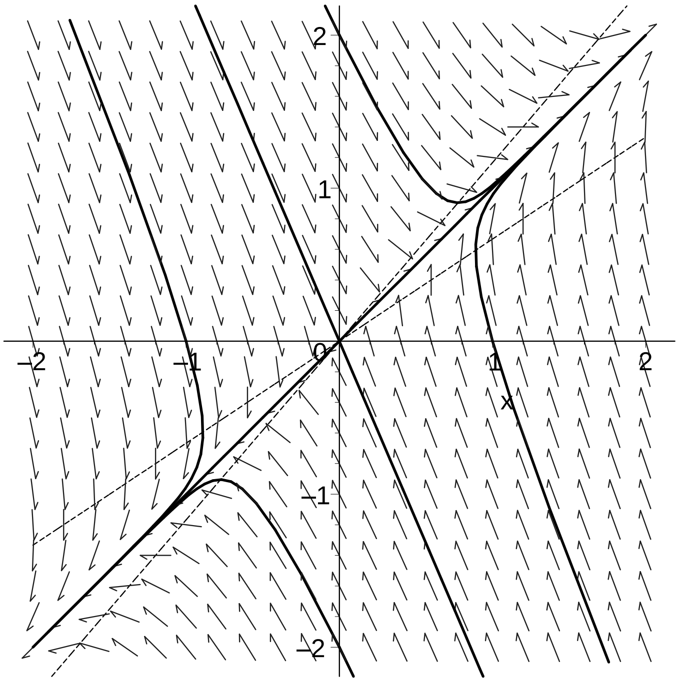
The characteristic polynomial is $\det(A-\lambda I) = \lambda^2 + 8\lambda - 9 = (\lambda+9)(\lambda-1)$, and solving $\det(A-\lambda I) = 0$ gives the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -9$. The equilibrium $(0,0)$ is a *saddle point*.

For $\lambda_1 = 1$, we have $A - \lambda_1 I = \begin{pmatrix} -3 & 3 \\ 7 & -7 \end{pmatrix}$, and solving $(A - \lambda_1 I)\mathbf{v} = \mathbf{0}$ gives $\mathbf{v} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, where c_1 is arbitrary. We can choose $c_1 = 1$ to get the eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = -9$, we have $A - \lambda_2 I = \begin{pmatrix} 7 & 3 \\ 7 & 3 \end{pmatrix}$, and solving $(A - \lambda_2 I)\mathbf{v} = \mathbf{0}$ gives $\mathbf{v} = c_2 \begin{pmatrix} -3 \\ 7 \end{pmatrix}$, where c_2 is arbitrary. We can choose $c_2 = 1$ to get the eigenvector $\mathbf{v}_2 = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$.

The vector field at $(1,0)$ is $\begin{pmatrix} -2 \\ 7 \end{pmatrix}$, and at $(0,1)$ it is $\begin{pmatrix} 3 \\ -6 \end{pmatrix}$. The vector field is vertical along the x-nullcline $-2x + 3y = 0$, and it is horizontal along the y-nullcline $7x - 6y = 0$.

Here is the phase portrait. The dashed lines are the nullclines.



7.5(c) The system is

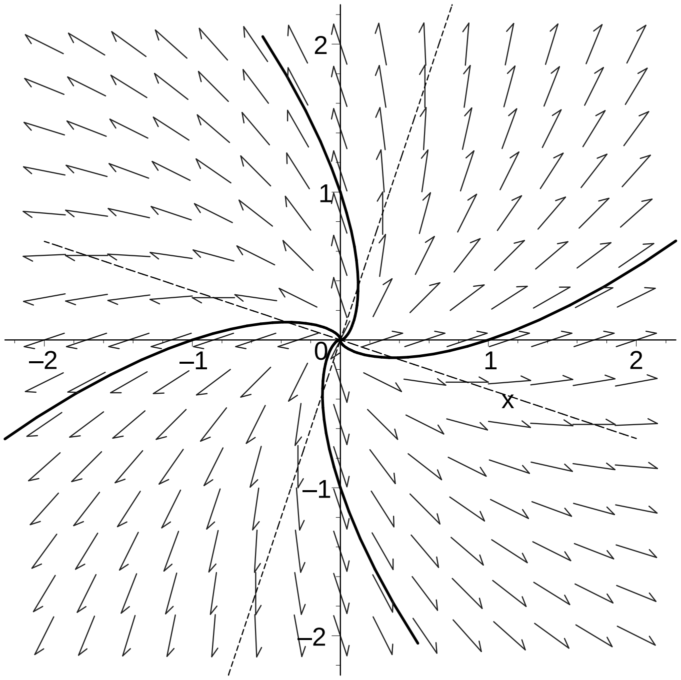
$$\begin{aligned}\dot{x} &= 3x - y\\ \dot{y} &= x + 3y\\ \dot{z} &= -2z\end{aligned}$$

The matrix of this system is

$$A = \begin{pmatrix} 3 & -1 & 0\\ 1 & 3 & 0\\ 0 & 0 & -2\end{pmatrix}.$$

This system is actually two decoupled subsystems, one involving x and y, and the other just z.

In the xy plane, we have the matrix $A_1 = \begin{pmatrix} 3 & -1\\ 1 & 3\end{pmatrix}$, which has eigenvalues $3 \pm i$. (These are also eigenvalues of A.) In the xy subsystem, we have an *spiral source*. The vector field at $(1, 0)$ is $\begin{pmatrix} 3\\ 1\end{pmatrix}$, and at $(0, 1)$ is $\begin{pmatrix} -1\\ 3\end{pmatrix}$. The x-nullcline is $3x - y = 0$, and the y-nullcline is $x + 3y = 0$. The vector field in the xy plane is



The third eigenvalue of A is -2, with eigenvector $\begin{pmatrix} 0\\ 0\\ 1\end{pmatrix}$. Solutions on the z axis approach the origin asymptotically.

Combining the two solutions, we see that the z coordinate of a typical solution decays to zero, while the (x, y) coordinates spiral outwards.

7.13(a) The system is

$$\dot{x} = 3x(1-x) - xy$$
$$\dot{y} = 5y(1-y) - 2xy$$

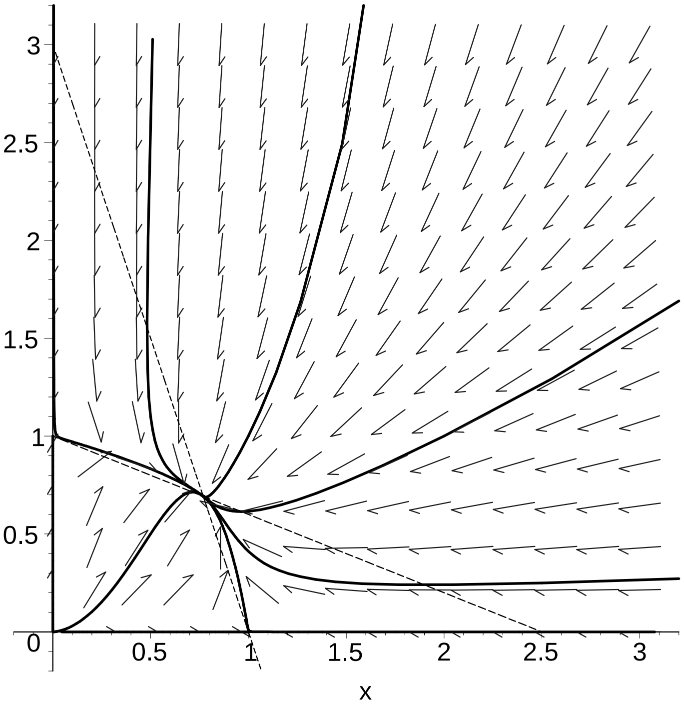
To find the x-nullclines, we solve

$$3x(1-x) - xy = 0 \quad \Longrightarrow \quad x(3-3x-y) = 0$$

so the x-nullclines are $x = 0$ and $y = 3 - 3x$. Similarly, we find the y-nullclines to be $y = 0$ and $y = -(2/5)x + 1$.

We find four equilibria: $(0,0)$, $(1,0)$, $(0,1)$ and $(10/13, 9/13)$. These are the points where an x-nullcline and y-nullcline intersect.

Here is the phase portrait.



Note that the nullclines separate the first quadrant into four regions.

- In the unbounded region, we have $\dot{x} < 0$ and $\dot{y} < 0$, so all solutions move towards the origin. Since the axes are themselves solutions, any solution in the unbounded region which is not on an axis must either converge to the equilibrium at $(10/13, 9/13)$, or cross a nullcline into one of the triangular regions.
- In the quadrilateral region with one corner at the origin, we have $\dot{x} > 0$ and $\dot{y} > 0$, so all solutions in this region move up and to the right, away from the origin. Any solution in this region that is not on an axis must either converge to the equilibrium $(10/13, 9/13)$, or cross a nullcline into one of the triangular regions.
- In the triangular region on the left, we have $\dot{x} > 0$ and $\dot{y} < 0$, so solutions in this region must move down and to the right. However, the lower boundary of this region is the y-nullcline, and on this boundary, the vector field points to the right, so solutions can not

leave this triangular region through the lower boundary. Similarly, the upper boundary is the x-nullcline, on which the vector field points down, so solutions can not leave the triangular region through this boundary. (This is an example of a *trapping region*: once a solution enters, it can never leave.) Thus, the only possible fate of trajectories in this region is to approach the equilibrium at the intersection of the boundaries. The intersection is the equilibrium point $(10/13, 9/13)$.

- In the triangular region on the right, we again have boundaries that are nullclines through which solutions can not leave. Since $\dot{x} < 0$ and $\dot{y} > 0$ in this region, all solutions must approach the equilibrium point $(10/13, 9/13)$.

Based on this nullcline analysis, we see that all solutions with $x(0) > 0$ and $y(0) > 0$ must converge to the equilibrium $(10/13, 9/13)$. If we interpret these equations as a population model, our conclusion is that both populations will survive.

The behavior near each of the equilibrium points can be seen in the phase portrait, but we can use the linearization at each equilibrium to verify the local behavior. The Jacobian at a point (x, y) is

$$J = \begin{pmatrix} 3 - 6x - y & -x \\ -2y & 5 - 2x - 10y \end{pmatrix}.$$

We find the following linearizations at each equilibrium:

- At $(0,0)$, $J = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, with eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 5$. Thus $(0,0)$ is a source.

- At $(1,0)$, $J = \begin{pmatrix} -3 & -1 \\ 0 & 3 \end{pmatrix}$, with eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 3$, so $(1,0)$ is a saddle point.

- At $(0,1)$, $J = \begin{pmatrix} 2 & 0 \\ -2 & -5 \end{pmatrix}$, with eigenvalues $\lambda_1 = -5$ and $\lambda_2 = 2$, so $(0,1)$ is also a saddle point.

- At $(10/13, 9/13)$, $J = \begin{pmatrix} -30/13 & -10/13 \\ -18/13 & -45/13 \end{pmatrix}$, with eigenvalues $\lambda_1 = (-75 - 3\sqrt{105})/26 \approx -4.07$ and $\lambda_2 = (-75 + 3\sqrt{105})/26 \approx -1.70$, so $(10/13, 9/13)$ is a sink.

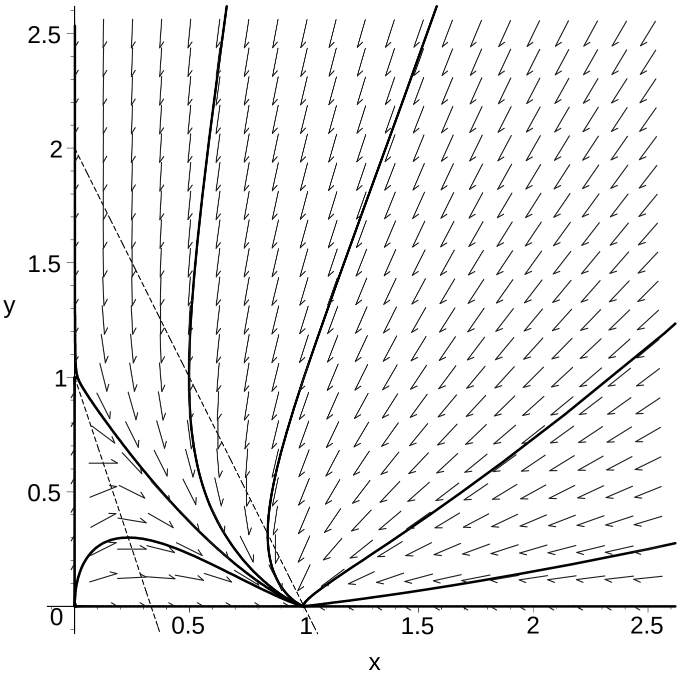
7.13(b) The system is

$$\begin{aligned} \dot{x} &= 2x(1 - x) - xy \\ \dot{y} &= y(1 - y) - 3xy \end{aligned}$$

The x-nullclines are $x = 0$ and $y = 2 - 2x$, and the y-nullclines are $y = 0$ and $y = 1 - 3x$.

There are four equilibria: $(0,0)$, $(1,0)$, $(0,1)$ and $(-1,4)$. The equilibrium at $(-1,4)$ is not relevant in a population model.

Here is the phase portrait:



The nullclines separate the first quadrant into three regions.

- In the unbounded region, $\dot{x} < 0$ and $\dot{y} < 0$, so all solutions must move down and to the left. This means all solutions must cross the x-nullcline $y = 2 - 2x$ and enter the middle region, or they must converge to the equilibrium at $(1, 0)$.
- In the lower left triangle with one corner at the origin, $\dot{x} > 0$ and $\dot{y} > 0$, so all solutions must move up and to the right. Thus all solution not on an axis must eventually cross the y-nullcline and enter the middle region.
- The middle quadrilateral region is a trapping region. Since the axes are also trajectories, solutions can not cross the axes. The lower left boundary is a y-nullcline, and the vector field on this nullcline points to the right, into the region. The upper right boundary is an x-nullcline, and the vector field on this line points down, into the region. So trajectories in this region can not leave. In this region, $\dot{x} > 0$ and $\dot{y} < 0$, so all solutions must move down and to the right. Thus all solutions must approach the equilibrium at $(1, 0)$.

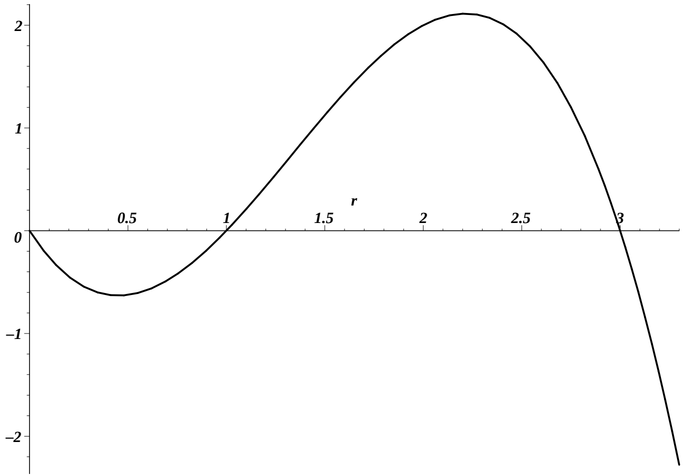
The conclusion is that all solutions with $x(0) > 0$ and $y(0) > 0$ converge to the point $(1, 0)$ asymptotically. If we interpret this system as a population model, we see that in the long term, the y population will die off and the x population will survive.

As in part (a), we could use the Jacobian matrix at each equilibrium to verify the behavior described above and shown in the phase portrait, but the problem doesn't actually require this. If you do so, you will find: $(0, 0)$ is a source, $(1, 0)$ is a sink, and $(0, 1)$ is a saddle.

8.1 *Short answer:* Theorem 8.3 says that any solution to $\dot{x} = f(x)$ must be monotonic, but $\sin(t)$ is not monotonic, so it can not be a solution to $\dot{x} = f(x)$.

Longer answer (based on the proof of Theorem 8.3): Consider, for example, the function $x(t) = \sin(t)$ at $t = 0$ and $t = \pi$. We have $x(0) = x(\pi) = 0$, so if $x(t)$ is a solution to $\dot{x} = f(x)$, then at $t = 0$ and $t = \pi$, the derivative of $x(t)$ must be $f(0)$; that is, $\dot{x}(0) = \dot{x}(\pi) = f(0)$. But we know that $\dot{x}(0) = \cos(0) = 1$ while $\dot{x}(\pi) = \cos(\pi) = -1$, which contradicts the previous observation, so $x(t) = \sin(t)$ can not be solution to $\dot{x} = f(x)$.

8.2 Consider just the equation for r. The graph of the right side of the equation (for $r \geq 0$) is



We see that $r = 0$ is a sink, $r = 1$ is a source, and $r = 3$ is a sink.

Now we interpret these in the xy plane: the origin is a spiral sink; the circle $r = 1$ is a periodic orbit, and in fact it is an *asymptotically stable limit cycle*; and the circle $r = 3$ is an unstable periodic orbit or *unstable limit cycle.*

Here are the ω-limit sets of the following points (r, θ):

- $(0, 0)$ is the origin, which is an equilibrium point. It's ω-limit set is $\boxed{\{(0, 0)\}}$.
- The trajectory beginning at $(r, \theta) = (1/2, 0)$ will spiral inwards and approach the origin asymptotically. Thus the ω-limit set of this trajectory is $\boxed{\text{the origin } (0, 0)}$.
- $(1, 0)$ is on the circle $r = 1$, which is a periodic orbit, so the ω-limit set is $\boxed{\text{the circle } r = 1}$.
- The trjactory beginning at $(r, \theta) = (2, 0)$ will spiral outwards and approach the circle $r = 3$ asymptotically, so the ω-limit set is $\boxed{\text{the circle } r = 3}$.

8.5

Figure 8.4(a):

α**-limit set**	**for**
$\{(0,0)\}$	$(0,0)$

ω**-limit set**	**for**
$\{(0,0)\}$	all points

Figure 8.4(b):

α**-limit set**	**for**
$\{(0,0)\}$	all points in the interior of the region enclosed by the periodic orbit
the periodic orbit	all points on the periodic orbit

ω**-limit set**	**for**
$\{(0,0)\}$	$(0,0)$
the periodic orbit	all points except the origin

Figure 8.4(c): Let S_1 and S_2 be the left and right spiral points, respectively, and let C_1 and C_2 be the left and right curves that connect P to itself, respectively.

α**-limit set**	**for**
P	$P \cup C_1 \cup C_2$
S_1	all points in the interior of the region enclosed by C_1
S_2	all points in the interior of the region enclosed by C_2

ω**-limit set**	**for**
P	$P \cup C_1 \cup C_2$
S_1	S_1
S_2	S_2
$P \cup C_1$	all points in the interior of the region enclosed by C_1 except S_1
$P \cup C_2$	all points in the interior of the region enclosed by C_2 except S_2
$P \cup C_1 \cup C_2$	all points "outside of" $P \cup C_1 \cup C_2$

Note that $P \cup C_1 \cup C_2$ is an ω-limit set, and the α- and ω-limit sets of points in $P \cup C_1 \cup C_2$ contain only the equilibrium point P. This is an example of case 3 of Theorem 8.8 (Poincaré-Bendixson).

T9.2 The Lorenz system is

$$\begin{aligned}\dot{x} &= \sigma(-x+y)\\ \dot{y} &= -xz+rx-y\\ \dot{z} &= xy-bz\end{aligned}$$

As explained on page 362, the parameters σ, b and r are parameters that correspond to physical properties of the original system that Lorenz was studying. It is therefore natural to assume that these parameters are all positive, and we will do so.

We must find x, y, and z where the vector field zero. The first equation implies $y = x$. Substituting this into the second and third equations and setting them to equal zero gives

$$\begin{aligned}x(-z+r-1) &= 0\\ x^2-bz &= 0\end{aligned}$$

The first of these implies $x = 0$ or $z = r-1$. If $x = 0$, the second implies $z = 0$, so we have an equilibrium at $(0,0,0)$. If $z = r-1$, the second equation implies $x = \pm\sqrt{b(r-1)}$. So if $r > 1$ (recall that we assumed $b > 0$), there are two more equilibria at

$$(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1) \quad \text{and} \quad (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1).$$

Computer Experiment 9.1

I used the Java applet at `http://www.cmp.caltech.edu/~mcc/chaos_new/Lorenz.html` to do this experiment. I picked several values of b, and for each value of b, I started with $\sigma = 10$ (or so), and then increased and decreased σ until I no longer observed a “butterfly”. The curves in the following plot show the results. It appears that for $1 < b < 3.5$, the system has a chaotic attractor for values of σ between the curves. For $b = 3.6$ or larger, I did not observe any chaotic attractors. I also tried some values of $b < 1$, but it was difficult to determine if the solution that I saw was a chaotic attractor, or a periodic orbit with high period.