Math 307 Supplemental Notes: Chaotic Attractors

Definition. Let f be a map, and x_0 an initial point. The **forward limit set** (or ω-**limit set**) of the orbit $\{f^n(x_0)\}$ is the set

$$\omega(x_0) = \{x : \text{for all } N > 0 \text{ and } \varepsilon > 0 \text{ there exists } n > N \text{ such that } |f^n(x_0) - x| < \varepsilon\}.$$

- This means that if $x \in \omega(x_0)$, the orbit of x_0 lands arbitrarily close to x infinitely many times.
- Suppose p is a *sink*, and x_0 is in the basin of p: $f^n(x_0) \to p$ as $n \to \infty$. Then $\omega(x_0) = \{p\}$. Also, $\omega(p) = p$.
- If p_1 is a period 2 point (with orbit $\{p_1, p_2\}$) that is a sink (of f^2), and x_0 is in the basin of p_1, then $\omega(x_0) = \{p_1, p_2\}$.
- Recall that the logistic map $G(x) = 4x(1-x)$ and the tent map $T_2(x)$ have *dense* orbits on $[0,1]$. The forward limit set of this orbit is the interval $[0,1]$.

Definition. Suppose the set S is a forward limit set (of some point x_0). We say an orbit $\{f^n(y_0)\}$ is **attracted to** S if $\omega(y_0)$ is contained in S.

- If p is a saddle point of f, then every point in the stable manifold of p is attracted to p. (Note that $\{p\}$ is a forward limit set, since $\omega(p) = \{p\}$.)

Definition. An **attractor** is a forward limit set S which attracts a set of nonzero measure (i.e. nonzero length, area or volume in $\mathbb{R}^1$, $\mathbb{R}^2$, or $\mathbb{R}^3$, respectively).

- A saddle is not an attractor.
- A sink is an attractor.
- The stable manifold of a saddle along with the saddle itself is *not* an attractor. It is not a forward limit set.

Definition. If $\{f^n(x_0)\}$ is a chaotic orbit, and $x_0 \in \omega(x_0)$, then $\omega(x_0)$ is called a **chaotic set**.

- For the logistic map G or the tent map T_2, the interval $[0,1]$ is a chaotic set. Why? The map has a dense chaotic orbit in $[0,1]$, and the forward limit set of this orbit is $[0,1]$. To satisfy the definition, let x_0 be any point in the dense chaotic orbit.

Definition. A **chaotic attractor** is a chaotic set that is also an attractor.