Math 307 Supplemental Notes: ω-limit Sets for Differential Equations

In the following, we will be referring to solutions to an autonomous system of differential equations

$$\dot{\mathbf{x}} = f(\mathbf{x}),$$

where $\mathbf{x} \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$. That is, f defines a **vector field** on $\mathbb{R}^n$, and in this context we refer to $\mathbb{R}^n$ as **phase space**.

Definition: The **flow** of the differential equation is the map $F(t, \mathbf{x}_0)$, which gives the points $\mathbf{x}(t)$ where $\mathbf{x}(t)$ is the solution for which $\mathbf{x}(0) = \mathbf{x}_0$.

Notes:

- For a fixed $\mathbf{x}_0$, $F(t, \mathbf{x}_0)$ gives the trajectory of the solution to the initial value problem $\mathbf{x}(0) = \mathbf{x}_0$.
- For a fixed t, $F(t, \mathbf{x})$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ that specifies where each initial condition $\mathbf{x}$ goes after time t. In Chapter 9, this point of view is emphasized by rewriting $F(t, \mathbf{x})$ as $F_t(\mathbf{x})$.
- In particular, the *time-1 map* is $F(1, \mathbf{x})$ or $F_1(\mathbf{x})$. This takes each point $\mathbf{x}$ to the position along its trajectory one time unit later.

Definition: The ω**-limit set** of a point $\mathbf{x}_0$ is the set

$$\omega(\mathbf{x}_0) = \{\mathbf{x} : \text{for all } T \text{ and all } \varepsilon > 0 \text{ there exists } t > T \text{ such that } |F(t, \mathbf{x}_0) - \mathbf{x}| < \varepsilon\}.$$

Equivalently,

$$\omega(\mathbf{x}_0) = \{\mathbf{x} : \text{there exists an unbounded, increasing sequence } \{t_k\} \text{ such that } \lim_{k\to\infty} F(t_k, \mathbf{x}_0) = \mathbf{x}\}.$$

Notes:

- If $\mathbf{x} \in \omega(\mathbf{x}_0)$, the trajectory through $\mathbf{x}_0$ passes arbitrarily close to $\mathbf{x}$ infinitely often as t increases.
- Suppose $\mathbf{p}$ is a *sink*, and $\mathbf{x}_0$ is in the basin of $\mathbf{p}$: $F(t, \mathbf{x}_0) \to \mathbf{p}$ as $t \to \infty$. Then $\omega(\mathbf{x}_0) = \{\mathbf{p}\}$. Also, $\omega(\mathbf{p}) = \{\mathbf{p}\}$.

Definition: The α**-limit set** of a point $\mathbf{x}_0$ is the set

$$\alpha(\mathbf{x}_0) = \{\mathbf{x} : \text{for all } T \text{ and all } \varepsilon > 0 \text{ there exists } t < T \text{ such that } |F(t, \mathbf{x}_0) - \mathbf{x}| < \varepsilon\}.$$

Equivalently,

$$\omega(\mathbf{x}_0) = \{\mathbf{x} : \text{there exists an unbounded, decreasing sequence } \{t_k\} \text{ such that } \lim_{k\to\infty} F(t_k, \mathbf{x}_0) = \mathbf{x}\}.$$

Notes:

- The α-limit sets are the ω-limit sets of the flow with time running backwards.
- If $\mathbf{x} \in \alpha(\mathbf{x}_0)$, the trajectory through $\mathbf{x}_0$ passes arbitrarily close to $\mathbf{x}$ infinitely often as t decreases.
- Any equilibrium point $\mathbf{p}$ is both an ω-limit set and an α-limit set: $\omega(\mathbf{p}) = \alpha(\mathbf{p}) = \{\mathbf{p}\}$.

Properties of ω-limit sets. The text discusses the following properties of ω-limit sets. For convenience, here are the properties that are also summarized on page 340:

1. **Existence:** The ω-limit set of a bounded orbit is non-empty.
2. **Closure:** An ω-limit set is closed.
3. **Invariance:** If $\mathbf{x} \in \omega(\mathbf{x}_0)$, then $F(t, \mathbf{x}) \in \omega(\mathbf{x}_0)$.
4. **Connectedness:** The ω-limit set of a bounded orbit is connected.
5. **Transitivity:** If $\mathbf{z} \in \omega(\mathbf{y})$ and $\mathbf{y} \in \omega(\mathbf{x})$, then $\mathbf{z} \in \omega(\mathbf{x})$.

Poincaré-Bendixson Theorem for Planar Vector Fields. (Theorem 8.8) Suppose the equilibria of a smooth vector field f in the plane are isolated, and let $F(t, \mathbf{x}_0)$ be bounded for $t \geq 0$. Then either

1. $\omega(\mathbf{x}_0)$ is an equilibrium, or
2. $\omega(\mathbf{x}_0)$ is a periodic orbit, or
3. for each $\mathbf{u}$ in $\omega(\mathbf{x}_0)$, the limit sets $\alpha(\mathbf{u})$ and $\omega(\mathbf{u})$ are equilibria.

Notes:

- This limits how "complicated" orbits of planar vector fields can be.
- In particular, there can be no chaotic orbits. (For differential equations, chaos can only occur in $\mathbb{R}^3$ or higher.)