Notes on the Periodically Forced Harmonic Oscillator

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Math 308 - Differential Equations

1 The Periodically Forced Harmonic Oscillator.

By *periodically forced harmonic oscillator*, we mean the linear second order nonhomogeneous differential equation

$$my'' + by' + ky = F\cos(\omega t) \tag{1}$$

where $m > 0$, $b \geq 0$, and $k > 0$. We can solve this problem completely; the goal of these notes is to study the behavior of the solutions, and to point out some special cases.

The parameter b is the ***damping coefficient*** (also known as the ***coefficient of friction***). We consider the cases $b = 0$ (undamped) and $b > 0$ (damped) separately.

2 Undamped ($b = 0$).

When $b = 0$, we have the equation

$$my'' + ky = F\cos(\omega t). \tag{2}$$

For convenience, define

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

This is the "natural frequency" of the undamped, unforced harmonic oscillator. To solve (2), we must consider two cases: $\omega \neq \omega_0$ and $\omega = \omega_0$.

2.1 $\omega \neq \omega_0$

By using the method of undetermined coefficients, we find the solution of (2) to be

$$y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F}{m(\omega^2 - \omega_0^2)}\cos(\omega t), \tag{3}$$

where, as usual, c_1 and c_2 are arbitrary constants.

Note that the amplitude of y_p becomes larger as ω approaches ω_0. This suggests that something other than a purely sinusoidal function may result when $\omega = \omega_0$.

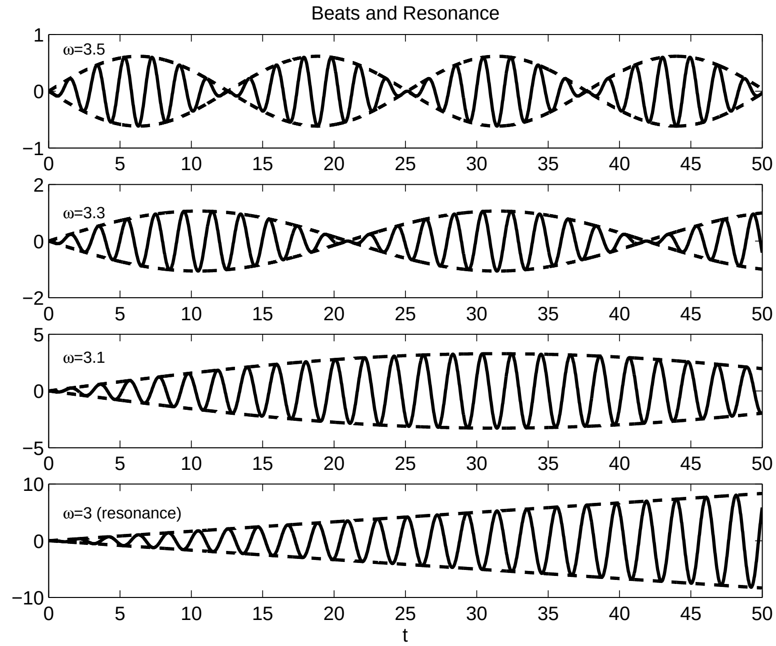

Figure 1: Solutions to (2) for several values of ω. The initial conditions are $y(0) = 0$ and $y'(0) = 0$. The solid curves are the actual solutions, while the dashed lines show the envelope (or modulation) of the amplitude.

Beats. Let us consider the initial conditions $y(0) = 0$ and $y'(0) = 0$. We must have $c_1 = -F/(m(\omega^2 - \omega_0^2))$ and $c_2 = 0$, which gives

$$y(t) = \frac{F}{m(\omega^2 - \omega_0^2)}(\cos(\omega t) - \cos(\omega_0 t)).$$

in (3). By using some trigonometric identities, we may rewrite this as

$$y(t) = \frac{2F}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{(\omega_0 + \omega)t}{2}\right) \sin\left(\frac{(\omega_0 - \omega)t}{2}\right). \tag{4}$$

Now, if $\omega \approx \omega_0$, we can think of this expression as the product of

$$\sin\left(\frac{(\omega_0 + \omega)t}{2}\right) \quad \text{and} \quad \frac{2F}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{(\omega_0 - \omega)t}{2}\right).$$

Since $\omega \approx \omega_0$, $|\omega_0 - \omega|$ is small, the first expression has a much higher frequency than the second. We see that the solution given in (4) is a "high" frequency oscillation, with an amplitude that is modulated by a low frequency oscillation. In Figure 1, we consider an example where $F = 1$, $m = 1$, and $\omega_0 = 3$. In the first three graphs, the solid lines are $y(t)$ given by (4), and the dashed lines show the *envelope* or *modulation* of the amplitude of the solution. Note that the vertical scale is different in each graph: the amplitude increases as ω approaches ω_0.

Remember that the solutions in Figure 1 are for the special initial conditions $y(0) = 0$ and $y'(0) = 0$. Beats occur with other initial conditions, but the function that modulates the amplitude of the high frequency oscillation will not be a simple sine function. Figure 2 shows a solution to (2) with $\omega_0 = 3.5$, but with $y(0) = 1$ and $y'(0) = -2$.

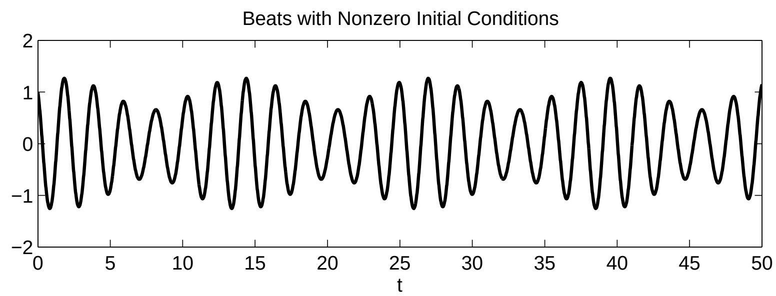

Figure 2: Solution to (2) for $\omega = 3.5$, with the initial conditions $y(0) = 1$ and $y'(0) = -2$.

2.2 $\omega = \omega_0$: Resonance

When $\omega = \omega_0$ in (2), the method of undetermined coefficients tells us that $y_p(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$ can not be the particular solution, because each term also solves the homogeneous equation. In this case, we multiply the guess by t to obtain $y_p(t) = At\cos(\omega_0 t) + Bt\sin(\omega_0 t)$. This means that the particular solution is *an oscillation whose amplitude grows linearly with t*. This phenomenon is known as ***resonance***. When we substitute y_p into (2) and simplify, we obtain

$$-2m\omega_0 A\sin(\omega_0 t) + 2m\omega_0 B\cos(\omega_0 t) = F\cos(\omega_0 t).$$

(The terms containing a factor of t automatically cancel.) For this equation to hold for all t, we must have $A = 0$ and $B = F/(2m\omega_0)$. Thus the general solution to (2) when $\omega = \omega_0$ is

$$y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F}{2m\omega_0}t\sin(\omega_0 t).$$

Special Case. For the special initial conditions $y(0) = 0$ and $y'(0) = 0$, we obtain $c_1 = c_2 = 0$, and so $y(t) = Ft\sin(\omega_0 t)/(2\omega_0)$. This solution is the fourth graph shown in Figure 1 (with $m = 1$ and $\omega_0 = 3$).

3 Damped ($b > 0$).

To solve (1) with the method of undetermined coefficients, we first guess $y_p(t) = A\cos(\omega t) + B\sin(\omega t)$. We then check that neither term in y_p solves the homogeneous equation. When $b > 0$, there are three possible forms for the homogeneous solution (underdamped, critically damped, and overdamped), but in all cases, the homogeneous solutions decay to zero as t increases, so neither term in y_p can be a solution to the homogeneous equation. So our first guess for y_p will work. Substituting this guess into (1) and choosing the coefficients to make y_p a solutions yields

$$y_p(t) = -\frac{(\omega^2 - \omega_0^2)Fm}{m^2(\omega^2 - \omega_0^2)^2 + b^2\omega^2}\cos(\omega t) + \frac{\omega b F}{m^2(\omega^2 - \omega_0^2)^2 + b^2\omega^2}\sin(\omega t) \tag{5}$$

Recall that an expression of the form $A\cos(\omega t) + B\sin(\omega t)$ may be written as $R\cos(\omega t - \phi)$, where $R = \sqrt{A^2 + B^2}$ and $\tan\phi = B/A$. (R is the ***amplitude*** and ϕ is the ***phase angle***.) With this, we have

$$y_p(t) = \frac{F}{\sqrt{m^2(\omega^2 - \omega_0^2)^2 + b^2\omega^2}}\cos(\omega t - \phi), \tag{6}$$

where

$$\phi = \arctan\left(\frac{-\omega b}{m(\omega^2 - \omega_0^2)}\right).$$

The general solution to (1) is the usual

$$y(t) = y_h(t) + y_p(t).$$

As mentioned earlier, when $b > 0$, the homogeneous solution will decay to zero as t increases. For this reason, the homogeneous solution is sometimes called the ***transient solution***, and y_p is called the ***steady state response***.

Example. Consider the initial value problem

$$y'' + 2y' + 10y = \cos(2t), \quad y(0) = -1/2, \quad y'(0) = 4.$$

We have $m = 1$, $b = 2$, $k = 10$, $F = 1$ and $\omega = 2$. We also find $\omega_0 = \sqrt{k/m} = \sqrt{10}$. The homogeneous solution is

$$y_h(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t),$$

and, using (5), we find the particular solution to be

$$y_p(t) = \frac{3}{26}\cos(2t) + \frac{1}{13}\sin(2t).$$

The solution to the initial value problem is

$$y(t) = -\frac{8}{13}e^{-t}\cos(3t) + \frac{14}{14}e^{-t}\sin(3t) + \frac{3}{26}\cos(2t) + \frac{1}{13}\sin(2t).$$

Figure 3 shows y_h, y_p and $y = y_h + y_p$.

3.1 Amplitude of the steady state.

Let us consider (6). The *amplitude* of the steady state is

$$\frac{F}{\sqrt{m^2(\omega^2 - \omega_0^2)^2 + b^2\omega^2}}.$$

We see that the amplitude depends on all the parameters in the differential equation. In particular, the amplitude of the steady state solution depends on the frequency ω of the forcing. Let's take $F = 1$, $m = 1$, $\omega_0 = 3$, so we have

$$\frac{1}{\sqrt{(\omega^2 - 9)^2 + b^2\omega^2}}.$$

The amplitude as a function of ω for several values of b is shown in Figure 4. Note that as b gets smaller, the peak in the response becomes larger, and the location of the peak approaches ω_0. The graph of the amplitude when $b = 0$, for which the response "blows up" at ω_0, is included for comparison.

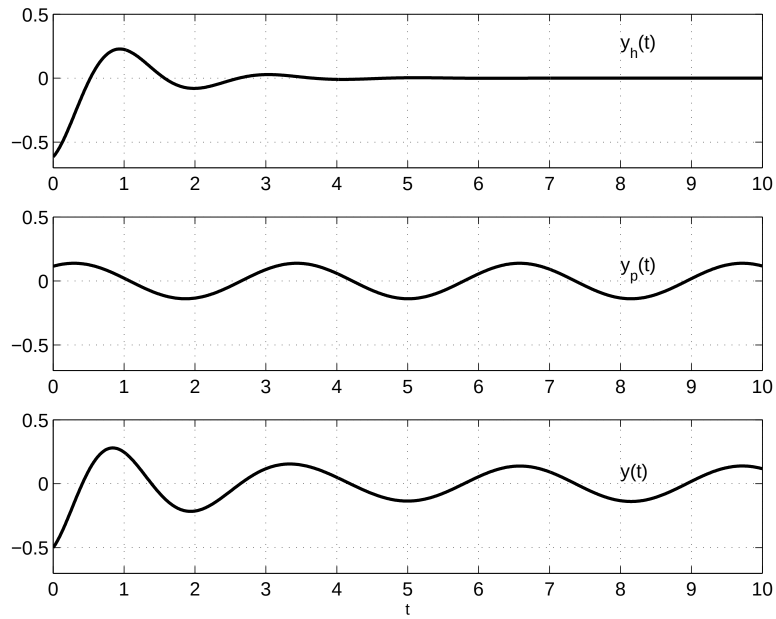

Figure 3: Graphs of y_h, y_p and the full solution to (1) with parameter values $m = 1$, $b = 2$, $k = 10$, $F = 1$, and with initial conditions $y(0) = -1/2$ and $y'(0) = 4$.

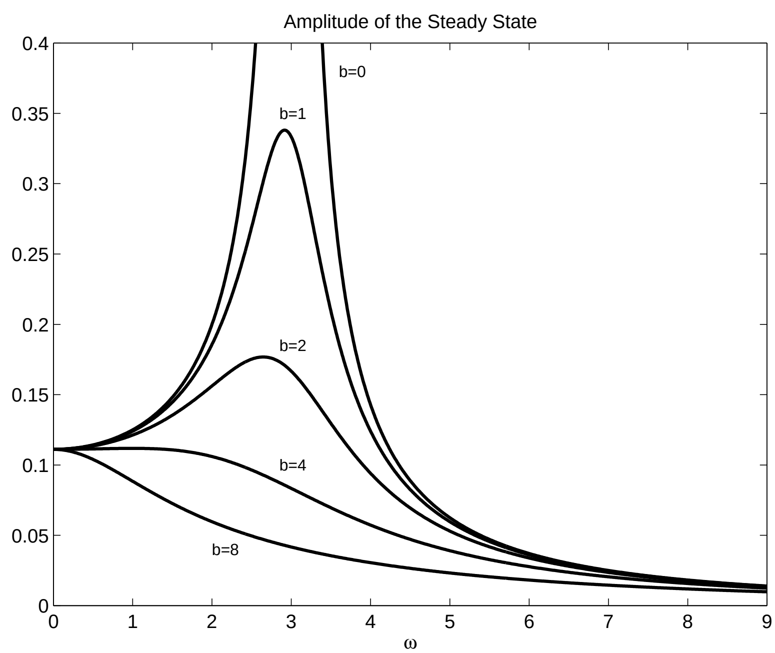

Figure 4: Amplitude of the steady state response as a function of ω for several values of b. The curve associated with $b = 0$ has a vertical asymptote at $\omega = 3$.

4 Brief Summary

1. The second order linear harmonic oscillator (damped or undamped) with sinusoidal forcing can be solved by using the method of undetermined coefficients.
2. In the undamped case, *beats* occur when the forcing frequency is close to (but not equal to) the natural frequency of the oscillator.
3. In the undamped case, *resonance* occurs when the forcing frequency is the same as the natural frequency of the oscillator.
4. In the damped case ($b > 0$), the homogeneous solution decays to zero as t increases, so the steady state behavior is determined by the particular solution.
5. In the damped case, the steady state behavior does not depend on the initial conditions.
6. The amplitude and phase of the steady state solution depend on all the parameters in the problem.

Words to Know: harmonic oscillator, damped, undamped, resonance, beats, transient, steady state, amplitude, phase