Math 311 **Exam 1** **Spring 2007**

- There are *six* problems–don't forget the second page when you print the exam!
- You must show your work to receive full credit. (You may use results that we have already derived without rederiving them.)
- Clearly indicate your final answer to each question. Erase or cross out anything that is incorrect or irrelevant. Multiple conflicting answers will receive no credit.
- You may refer to the text, your lecture notes, and the homework solutions while doing the exam. You may not use any other material.
- In your answer sheets, use just one side of each page. Start each problem on a new page. (Sub-parts can be on the same page.)

1. Let $f(x) = x^2$ on the interval $0 \le x \le 2$.
   (a) On the interval $-6 \le x \le 6$, sketch the function to which the Fourier sine series of $f(x)$ converges.
   (b) Repeat (a), but for the Fourier cosine series.

2. Find the Fourier cosine series of

$$f(x) = \begin{cases} 1-x & 0 \le x \le 1 \\ 0 & 1 \le x \le 3 \end{cases} \tag{1}$$

   (Note that f is defined on $0 \le x \le 3$, so $L = 3$.)

3. Find the eigenvalues and the corresponding eigenfunctions for the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + 2\frac{d\phi}{dx} = -\lambda\phi, \quad \phi(0) = 0, \quad \phi(L) = 0, \tag{2}$$

   where $L > 0$ is a constant. You may assume that λ is real. (Be careful! In your quest for eigenvalues, you will have to consider three cases separately, but these three cases are *not* $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$.)

4. Suppose $u(x,t)$ satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} - u \tag{3}$$

   with the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0 \tag{4}$$

   where $L > 0$ is a constant.

   If the method of separation of variables is used to solve this problem, what are the two ordinary differential equations that result, and what (if any) are the boundary conditions for them? Which one is an eigenvalue problem?

5. Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin x, \quad 0 < x < \pi, \tag{5}$$

with boundary conditions

$$u(0,t) = 0, \quad u(\pi,t) = \beta, \quad t > 0 \tag{6}$$

and initial condition

$$u(x,0) = 0, \quad 0 < x < \pi. \tag{7}$$

where β is a constant.

(a) Find the equilibrium solution $u_e(x)$.

(b) Suppose that $u(x,t)$ solves the given PDE, BCs and IC. Let $w(x,t) = u(x,t) - u_e(x)$. Find the PDE, BCs, and IC satisfied by w, and solve for $w(x,t)$. (Your answer will be in the form of an infinite series. Be sure that you have given the formula for any coefficients in your answer; that is, do the integrals!)

(Having completed (a) and (b), you have solved the original problem, since $u(x,t) = u_e(x) + w(x,t)$ is the solution.)

6. Solve Laplace's equation inside the semicircle of radius a ($0 < r < a$, $0 < \theta < \pi$), subject to the boundary conditions that $\frac{\partial u}{\partial r}(a,\theta) = 1$ on the circular boundary, and $u = 0$ on the x axis.

(As in the previous problem, find explicit formulas for any coefficients in your series solution–do the integrals!)