Math 311 **Exam 2** **Spring 2007**

- There are *five* problems–don't forget the second page when you print the exam!
- You must show your work to receive full credit. (You may use results that we have already derived without rederiving them.)
- Clearly indicate your final answer to each question. Erase or cross out anything that is incorrect or irrelevant. Multiple conflicting answers will receive no credit.
- You may refer to the text, your lecture notes, and the homework solutions while doing the exam. You may not use any other material.
- In your answer sheets, use just one side of each page. Start each problem on a new page. (Sub-parts can be on the same page.)

1. (24 pts.) Solve the wave equation on the interval $0 \leq x \leq L$ with "free ends"

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0$$

with initial conditions

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = g(x)$$

2. (14 pts.) Consider the equation for a damped vibrating string:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$$

with the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. The damping coefficient β is positive ($\beta > 0$).

The total energy is the sum of the kinetic and potential energies:

$$E = \frac{1}{2} \int_0^L \left(\frac{\partial u}{\partial t} \right)^2 dx + \frac{c^2}{2} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx$$

Show that the energy is a decreasing function of time.

3. (14 pts.) These questions concern the vibrating rectangular membrane (section 7.3).

When the rectangle is not square (i.e. if $L \neq H$), assume (without loss of generality) that $L > H$.

   (a) What are the two lowest frequencies of vibration of the rectangular membrane?

   (b) Is it possible to adjust the dimensions L and H of the membrane so that the second lowest frequency is twice the lowest frequency? (For those of you who are musically inclined, this means that the two lowest frequencies are an octave apart.)

(c) Is it possible to adjust the dimensions so that the second lowest frequency is exactly 1.5 times the lowest? (For the musicians: the interval corresponds to a *fifth*.)

4. (24 pts.) Consider the Sturm-Liouville problem

$$\frac{d}{dx}\left(e^x \frac{d\phi}{dx}\right) + \lambda(1+x)\phi = 0, \quad \phi(0) = 0, \quad \frac{d\phi}{dx}(1) = 0.$$

(a) Show that all the eigenvalues are positive. (Be sure to include an explanation of why $\lambda = 0$ is not an eigenvalue.)

(b) Find upper and (nonzero) lower bounds for the first eigenvalue.

(c) Suppose the eigenfunctions are $\phi_n(x)$, $n = 1, 2, 3, \ldots$. The theorem on the completeness of the eigenfunctions says that we may represent a function $f(x)$ as

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

Give the formula for a_n in this case. (Express your answer with appropriate integrals.)

5. (24 pts.) Solve the heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

inside a circle of radius a with an insulated boundary at $r = a$, and with initial condition

$$u(r, \theta, 0) = f(r, \theta).$$

Give the formulas for the coefficients in your answer in terms of iterated double integrals over the domain (i.e. your integrals will have $dr\,d\theta$ or $d\theta\,dr$, not dA). Clearly define any symbols or notation that you use in your solution.