Linearization

The text focuses on linear partial differential equations. PDEs that arise from realistic models of the natural world are generally *nonlinear*. The study of linear PDEs is still useful, because often the solutions to a nonlinear PDE can be approximated by the solutions to an associated linear PDE. In this module, we discuss the *linearization* of a nonlinear PDE about a known solution.

We will use examples with one space dimension (so our solutions are $u(x,t)$), but the same idea applies to higher dimensional problems.

Example 1. Consider the nonlinear PDE for $u(x,t)$

$$\frac{\partial u}{\partial t} = u(1-u) + \frac{\partial^2 u}{\partial x^2} \quad (0 < x < L) \tag{1}$$

with boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0 \tag{2}$$

On the right of (1), we have the term $u(1-u) = u - u^2$, and the term u^2 makes this a nonlinear PDE.

The PDE has two uniform equilibrium solutions, $u(x,t) = 0$ and $u(x,t) = 1$. (These are easy enough to find: *equilibrium* means $\frac{\partial u}{\partial t} = 0$, and *uniform* means $\frac{\partial^2 u}{\partial x^2} = 0$, so the uniform equilibrium solutions are the solutions to $u(1-u) = 0$. The constant solutions $u = 0$ and $u = 1$ also satisfy the boundary conditions, so these are the uniform equilibrium solutions to the PDE.)

Now suppose that the initial conditions to the problem are such that $u(x,t)$ is initially close to $u(x,t) = 0$. That is, we consider $u(x,t) = 0 + \varepsilon w(x,t)$, where ε is "small". If we make this substitution into the PDE, we obtain

$$\varepsilon \frac{\partial w}{\partial t} = (\varepsilon w)(1 - \varepsilon w) + \varepsilon \frac{\partial^2 w}{\partial x^2}$$

or

$$\frac{\partial w}{\partial t} = w - \varepsilon w^2 + \frac{\partial^2 w}{\partial x^2}$$

Now take the limit $\varepsilon \to 0$; we obtain the *linearization of* (1) *at* $u = 0$:

$$\frac{\partial w}{\partial t} = w + \frac{\partial^2 w}{\partial x^2} \tag{3}$$

Also, since $u(x,t) = \varepsilon w(x,t)$, w satisfies the same boundary conditions as u.

We can solve the linearization by the method of separation of variables. If $u(x,t) = \phi(x)h(t)$, then h satisfies

$$\frac{dh}{dt} = -\lambda h,$$

and ϕ satisfies the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \phi = -\lambda \phi, \quad \frac{d\phi}{dx}(0) = 0, \ \frac{d\phi}{dx}(L) = 0.$$

The eigenvalues are $\lambda_0 = -1$ (with eigenfunction $\phi_0(x) = 1$), and $\lambda_n = (n\pi/L)^2 - 1$ (with eigenfunction $\phi_n(x) = \cos(n\pi x/L)$ for $n = 1,2,3,\ldots$. The solution to the equation for h is $h_0 e^{-\lambda t}$. We form the general solution as the infinite series

$$w(x,t) = c_0 e^t + \sum_{n=1}^{\infty} c_n e^{-((n\pi/L)^2-1)t} \cos(n\pi x/L)$$

The coefficients c_n ($n = 0,1,2,\ldots$) depend on the initial conditions. Unless we choose the initial conditions very carefully, we expect that each coefficient is nonzero. In particular, $c_0 \neq 0$. The term $c_0 e^t$ will grow exponentially as t increases. Moreover, if $L > \pi$, then $\lambda_1 = (\pi/L)^2 - 1 < 0$, and there will also be exponential growth of the term $c_1 e^{-((\pi/L)^2-1)t} \cos(\pi x/L)$. In general, if $L > m\pi$, then the "modes" associated with λ_n for $n = 0,1,2,\ldots,m$ will all grow exponentially as t increases.

Remember that εw represents a "small" perturbation from the equilibrium $u = 0$. The solution to the linearization at $u = 0$ tells us that this small perturbation will not remain small; instead, it will grow exponentially. We say that the equilibrium $u = 0$ is *unstable*.

Be careful to interpret this result appropriately. We are not saying that $u(x,t)$ will continue to grow exponentially for an arbitrarily long time. The linearization was based on the assumption that the perturbation was small. If the solution to the linearized equation grows exponentially, it will not remain small–but then the linearization is no longer a valid approximation. Our analysis so far does not tell us what will happen to $u(x,t)$ in the long run; it just tells us that if the solution starts near $u = 0$, it will not remain close to $u = 0$.

Now let's consider the linearization at the other uniform equilibrium, $u = 1$. We let $u(x,t) = 1 + \varepsilon w(x,t)$; the PDE becomes

$$\frac{\partial w}{\partial t} = -w - \varepsilon w^2 + \frac{\partial^2 w}{\partial x^2}$$

and $\varepsilon = 0$ gives

$$\frac{\partial w}{\partial t} = -w + \frac{\partial^2 w}{\partial x^2}$$

In this case, the eigenvalues are $\lambda_0 = 1$, and $\lambda_n = (n\pi/L)^2 + 1$ ($n = 1,2,3,\ldots$), with the same eigenfunctions as before. The general solution is

$$w(x,t) = c_0 e^{-t} + \sum_{n=1}^{\infty} c_n e^{-((n\pi/L)^2+1)t} \cos(n\pi x/L)$$

All the exponential terms have negative coefficients in the exponents, so all the modes decay to zero.

It appears that a small perturbation to the equilibrium $u = 1$ will remain small. (In fact, the perturbation will decay to zero.) In this case, we say that the equilibrium is *stable*.

Linearization: A General Procedure

The steps to derive the linearization of the PDE in the previous example will not work in general. In that example, the nonlinear term was u^2, a polynomial, and we could obtain the linearization by using a little algebra and then setting $\varepsilon = 0$. This will not work with a nonlinearity such as $\sin(u)$. Here we give a general procedure for obtaining the linearization at a known uniform equilibrium solution. (The method also works for nonuniform solutions, but we will not see any examples here.)

Linearization Procedure
Suppose $u(x,t) = U_0$ is an uniform equilibrium solution to the PDE.

1. Substitute $u(x,t) = U_0 + \varepsilon w(x,t)$ into the PDE.
2. Take the derivative of all expressions in the PDE *with respect to* ε. (Don't forget to use the chain rule where necessary.)
3. Set $\varepsilon = 0$. The remaining equation is the linearization at $u(x,t) = U_0$.

Example 2. Consider

$$\frac{\partial^2 u}{\partial t^2} + \sin u = \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions

$$u(0,t) = 0,\ u(L,t) = 0.$$

It is easily verifed that $u(x,t) = U_0 = 0$ is an equilibrium solution. To find the linearization of the PDE, we set $u(x,t) = 0 + \varepsilon w(x,t)$ and obtain

$$\varepsilon \frac{\partial^2 w}{\partial t^2} + \sin(\varepsilon w) = \varepsilon \frac{\partial^2 w}{\partial x^2}$$

Then we take the derivative with respect to ε. (For example, $\frac{\partial}{\partial \varepsilon}\left(\varepsilon \frac{\partial^2 w}{\partial t^2}\right) = \frac{\partial^2 w}{\partial t^2}$.) The equation becomes

$$\frac{\partial^2 w}{\partial t^2} + \cos(\varepsilon w) w = \frac{\partial^2 w}{\partial x^2}$$

(Note how the chain rule was used: $\frac{\partial}{\partial \varepsilon}\sin(\varepsilon w) = \cos(\varepsilon w) w$.) Finally, we set $\varepsilon = 0$ to obtain the linearization:

$$\frac{\partial^2 w}{\partial t^2} + w = \frac{\partial^2 w}{\partial x^2}$$

(since $\cos(0) = 1$).

Exercises

1. Apply the linearization procedue to the first example to rederive the linearization at $u = 0$ and at $u = 1$. (Just rederive the equations; don't repeat the stability analysis.)
2. Consider the PDE and boundary conditions

$$\frac{\partial^2 u}{\partial t^2} + \sin u = \frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0.$$

   Find all the uniform equilibrium solutions, and use the linearization at each equilibrium to determine whether or not it is stable.