1.2.3 With no sources, but with variable cross section $A(x)$, equation (1.2.4) is

$$\frac{d}{dt} \int_{0}^{L} e(x,t)A(x) \, dx = \phi(a,t)A(a) - \phi(b,t)A(b) = - \int_{a}^{b} \frac{\partial}{\partial x} (\phi(x,t)A(x)) \, dx$$

(1)

By moving the $t$ derivative inside the integral, we can combine the integrals to obtain

$$\int_{a}^{b} \left[ \frac{\partial e}{\partial t}(x,t)A(x) + \frac{\partial}{\partial x} (\phi(x,t)A(x)) \right] \, dx = 0$$

(2)

Since $a$ and $b$ were arbitrary, this implies

$$\frac{\partial e}{\partial t}(x,t)A(x) + \frac{\partial}{\partial x} (\phi(x,t)A(x)) = 0$$

(3)

The problem tells us to assume constant thermal properties, so $c$, $\rho$ and $K_0$ are constants. Fourier’s Law still holds:

$$\phi(x,t) = -K_0 \frac{\partial u}{\partial x}$$

(4)

By setting $e = c\rho u$ and using Fourier’s Law, (3) becomes

$$c\rho A(x) \frac{\partial u}{\partial t} - K_0 \frac{\partial}{\partial x} \left( A(x) \frac{\partial u}{\partial x} \right) = 0$$

(5)

or

$$\frac{\partial u}{\partial t} = \frac{k}{A(x)} \frac{\partial}{\partial x} \left( A(x) \frac{\partial u}{\partial x} \right)$$

(6)

where $k = K_0/(c\rho)$.
1.4.7(a) To find an equilibrium solution \( u_e(x) \), we must solve

\[
\frac{d^2 u_e}{dx^2} + 1 = 0, \quad \frac{du_e}{dx}(0) = 1, \quad \frac{du_e}{dx}(L) = \beta. \tag{7}
\]

We can simply integrate to find

\[
\frac{du_e}{dx} = -x + c_1, \quad \text{and} \quad u_e(x) = -\frac{x^2}{2} + c_1 x + c_2. \tag{8}
\]

At \( x = 0 \) we must have \( \frac{du_e}{dx}(0) = c_1 = 1 \). At \( x = L \) we must have \( \frac{du_e}{dx}(L) = -L + 1 = \beta \). So we only have an equilibrium solution if

\[
\beta = 1 - L. \tag{9}
\]

The equilibrium solution is

\[
u_e(x) = -\frac{x^2}{2} + x + c_2\tag{10}\]

where \( c_2 \) depends on the initial conditions.

To find \( c_2 \), we integrate the PDE with respect to \( x \) from 0 to \( L \) and take the \( t \) derivative outside the integral on the left to obtain

\[
\frac{d}{dt} \int_0^L u(x,t) \, dx = \frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t) + L
\]

\[
= \beta - 1 + L
\]

\[
= 0 \tag{11}
\]

This says that the quantity \( \int_0^L u(x,t) \, dx \) remains constant for all \( t \). In particular,

\[
\int_0^L u(x,0) \, dx = \int_0^L u_e(x) \, dx \tag{12}
\]

On the left of this equation we have

\[
\int_0^L u(x,0) \, dx = \int_0^L f(x) \, dx, \tag{13}
\]

and on the right we have

\[
\int_0^L u_e(x) \, dx = \int_0^L \left( -\frac{x^2}{2} + x + c_2 \right) \, dx = -\frac{L^3}{6} + \frac{L^2}{2} + c_2 L \tag{14}
\]

So

\[
-\frac{L^3}{6} + \frac{L^2}{2} + c_2 L = \int_0^L f(x) \, dx \tag{15}
\]

and solving for \( c_2 \) gives

\[
c_2 = \frac{L^2}{6} - \frac{L}{2} + \frac{1}{L} \int_0^L f(x) \, dx \tag{16}
\]
Physical Interpretation.

Brief explanation: The condition $\beta = 1 - L$ ensures that the rate of heat loss through the boundaries exactly equals the rate of heat production in the interval $0 < x < L$.

Longer discussion:

The PDE is the heat equation (with $K = c\rho$) with a constant source $Q = c\rho$, so heat is being generated at each $x$ in the interval $0 < x < L$. The total rate at which heat is generated in the interval is $A \int_0^L Q(x,t) \, dx = Ac\rho L$.

At the left end, we have $\frac{\partial u}{\partial x}(0,t) = 1$. From Fourier’s Law, we know that $\phi = -K \frac{\partial u}{\partial x}$, so the left boundary condition says heat flow rate through the left end is $-AK_0 = -Ac\rho$ (since $K_0 = c\rho$). Heat is also being forced to move through the right end at a constant rate, with the total heat flow through the right being $-Ac\rho\beta$. (The direction depends on the sign of $\beta$. If $\beta > 0$, heat is flowing into the region, while if $\beta < 0$, heat is flowing out of the region.) The net rate of heat flow out of the region is $Ac\rho - Ac\rho\beta$.

In order to have an equilibrium solution, the rate at which heat is generated within the interval $0 < x < L$ must equal the rate at which heat is removed through the boundaries. Thus we must have

$$Ac\rho - Ac\rho\beta = Ac\rho L$$

or

$$1 - \beta = L.$$