Math 311 Applied Mathematics - Physical Sciences **Spring 2007**

Homework 1 Selected Solutions

1.2.3 With no sources, but with variable cross section $A(x)$, equation (1.2.4) is

$$\frac{d}{dt}\int_0^L e(x,t)A(x)\,dx = \phi(a,t)A(a) - \phi(b,t)A(b) = -\int_a^b \frac{\partial}{\partial x}\left(\phi(x,t)A(x)\right)\,dx \tag{1}$$

By moving the t derivative inside the integral, we can combine the integrals to obtain

$$\int_a^b \left[\frac{\partial e}{\partial t}(x,t)A(x) + \frac{\partial}{\partial x}\left(\phi(x,t)A(x)\right)\right]\,dx = 0 \tag{2}$$

Since a and b were arbitrary, this implies

$$\frac{\partial e}{\partial t}(x,t)A(x) + \frac{\partial}{\partial x}\left(\phi(x,t)A(x)\right) = 0 \tag{3}$$

The problem tells us to assume constant thermal properties, so c, ρ and K_0 are constants. Fourier's Law still holds:

$$\phi(x,t) = -K_0\frac{\partial u}{\partial x} \tag{4}$$

By setting $e = c\rho u$ and using Fourier's Law, (3) becomes

$$c\rho A(x)\frac{\partial u}{\partial t} - K_0\frac{\partial}{\partial x}\left(A(x)\frac{\partial u}{\partial x}\right) = 0 \tag{5}$$

or

$$\frac{\partial u}{\partial t} = \frac{k}{A(x)}\frac{\partial}{\partial x}\left(A(x)\frac{\partial u}{\partial x}\right) \tag{6}$$

where $k = K_0/(c\rho)$.

1.4.7(a) To find an equilibrium solution $u_e(x)$, we must solve

$$\frac{d^2u_e}{dx^2} + 1 = 0, \quad \frac{du_e}{dx}(0) = 1, \quad \frac{du_e}{dx}(L) = \beta. \tag{7}$$

We can simply integrate to find

$$\frac{du_e}{dx} = -x + c_1, \quad \text{and} \quad u_e(x) = -\frac{x^2}{2} + c_1 x + c_2. \tag{8}$$

At $x = 0$ we must have $\frac{du_e}{dx}(0) = c_1 = 1$. At $x = L$ we must have $\frac{du_e}{dx}(L) = -L + 1 = \beta$. So we only have an equilibrium solution if

$$\beta = 1 - L \tag{9}$$

The equilibrium solution is

$$u_e(x) = -\frac{x^2}{2} + x + c_2 \tag{10}$$

where c_2 depends on the initial conditions.

To find c_2, we integrate the PDE with respect to x from 0 to L and take the t derivative outside the integral on the left to obtain

$$\begin{aligned}\frac{d}{dt}\int_0^L u(x,t)\,dx &= \frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t) + L \\ &= \beta - 1 + L \\ &= 0\end{aligned} \tag{11}$$

This says that the quantity $\int_0^L u(x,t)\,dx$ remains constant for all t. In particular,

$$\int_0^L u(x,0)\,dx = \int_0^L u_e(x)\,dx \tag{12}$$

On the left of this equation we have

$$\int_0^L u(x,0)\,dx = \int_0^L f(x)\,dx, \tag{13}$$

and on the right we have

$$\int_0^L u_e(x)\,dx = \int_0^L \left(-\frac{x^2}{2} + x + c_2\right)dx = -\frac{L^3}{6} + \frac{L^2}{2} + c_2 L \tag{14}$$

So

$$-\frac{L^3}{6} + \frac{L^2}{2} + c_2 L = \int_0^L f(x)\,dx \tag{15}$$

and solving for c_2 gives

$$c_2 = \frac{L^2}{6} - \frac{L}{2} + \frac{1}{L}\int_0^L f(x)\,dx \tag{16}$$

Physical Interpretation.

Brief explanation: The condition $\beta = 1 - L$ ensures that the rate of heat loss through the boundaries exactly equals the rate of heat production in the interval $0 < x < L$.

Longer discussion:

The PDE is the heat equation (with $K_0 = c\rho$) with a constant source $Q = c\rho$, so heat is being generated at each x in the interval $0 < x < L$. The total rate at which heat is generated in the interval is $A\int_0^L Q(x,t)\,dx = Ac\rho L$.

At the left end, we have $\frac{\partial u}{\partial x}(0,t) = 1$. From Fourier's Law, we know that $\phi = -K_0\frac{\partial u}{\partial x}$, so the left boundary condition says heat flow rate through the left end is $-AK_0 = -Ac\rho$ (since $K_0 = c\rho$). Heat is also being forced to move through the right end at a constant rate, with the total heat flow through the right being $-Ac\rho\beta$. (The direction depends on the sign of β. If $\beta > 0$, heat is flowing into the region, while if $\beta < 0$, heat is flowing out of the region.) The net rate of heat flow *out* of the region is $Ac\rho - Ac\rho\beta$.

In order to have an equilibrium solution, the rate at which heat is generated within the interval $0 < x < L$ must equal the rate at which heat is removed through the boundaries. Thus we must have

$$Ac\rho - Ac\rho\beta = Ac\rho L \tag{17}$$

or

$$1 - \beta = L. \tag{18}$$