Math 311 Applied Mathematics - Physical Sciences **Spring 2007**

Homework 2 Selected Solutions

2.3.8 The PDE is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u \tag{1}$$

where $\alpha > 0$, with boundary conditions

$$u(0,t) = 0, \quad \text{and} \quad u(L,t) = 0. \tag{2}$$

(a) The equation for an equilibrium is

$$k\frac{\partial^2 u}{\partial x^2} - \alpha u = 0 \tag{3}$$

This second order ordinary differential equation has the general solution

$$u_e(x) = c_1 e^{\sqrt{\alpha/k}\,x} + c_2 e^{-\sqrt{\alpha/k}\,x} \tag{4}$$

Alternatively, we can use sinh and cosh to express the fundamental set of solutions; then the general solution may be written

$$u_e(x) = c_1 \cosh\left(\sqrt{\alpha/k}\,x\right) + c_2 \sinh\left(\sqrt{\alpha/k}\,x\right) \tag{5}$$

Regardless of the form that we choose, if we try to satisfy the boundary conditions $u_e(0) = 0$ and $u_e(L) = 0$, we find $c_1 = 0$ and $c_2 = 0$. So the only possible equilibrium solution is

$$u_e(x) = 0. \tag{6}$$

(b) To solve the time-dependent problem, we begin with separation of variables:

$$u(x,t) = \phi(x)h(t) \tag{7}$$

Then the PDE becomes

$$\phi\frac{dh}{dt} = k\frac{d^2\phi}{dx^2}h - \alpha\phi h \tag{8}$$

and the boundary conditions become

$$\phi(0) = 0 \quad \text{and} \quad \phi(L) = 0. \tag{9}$$

After some algebra, (8) leads to

$$\frac{1}{h}\frac{dh}{dt} = \frac{1}{\phi}\left(k\frac{d^2\phi}{dx^2} - \alpha\phi\right) = -\lambda \tag{10}$$

where I have introduced the separation constant λ. Thus h must satisfy the ordinary differential equation

$$\frac{dh}{dt} = -\lambda h, \tag{11}$$

and ϕ must satisfy the eigenvalue problem

$$k\frac{d^2\phi}{dx^2} - (\alpha - \lambda)\phi = 0, \quad \phi(0) = 0, \quad \phi(L) = 0. \tag{12}$$

We now solve this eigenvalue problem. The form of the general solution to the differential equation in (12) depends on the sign of $\alpha - \lambda$. We have three cases to consider.

$\boldsymbol{\lambda < \alpha}$: In this case, $\alpha - \lambda > 0$, and the general solution to the ordinary differential equation in the eigenvalue problem is

$$\phi(x) = c_1 \cosh(\mu x) + c_2 \sinh(\mu x) \tag{13}$$

where, for convenience, I have introduced

$$\mu = \sqrt{\frac{\alpha - \lambda}{k}} \tag{14}$$

If we now try to satisfy the boundary conditions of the eigenvalue problem, we find $c_1 = 0$ and $c_2 = 0$; the only solution to the eigenvalue problem when $\lambda < \alpha$ is the trivial solution. Thus there are no eigenvalues λ such that $\lambda < \alpha$.

$\boldsymbol{\lambda = \alpha}$: In this case, the general solution to the differential equation in the eigenvalue problem is

$$\phi(x) = c_1 x + c_2, \tag{15}$$

and the only solution that satisfies the boundary conditions is the trivial solution $\phi(x) = 0$. So $\lambda = \alpha$ is not an eigenvalue.

$\boldsymbol{\lambda > \alpha}$: For convenience, let

$$\omega = \sqrt{\frac{\lambda - \alpha}{k}} \tag{16}$$

The general solution to the differential equation of the eigenvalue problem (12) is

$$\phi(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x) \tag{17}$$

The first boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The second boundary condition implies

$$c_2 \sin(\omega L) = 0 \tag{18}$$

Since $c_2 = 0$ only gives us the trivial solution, we assume $c_2 \neq 0$. Then $\sin(\omega L) = 0$ implies

$$\omega L = n\pi, \quad n \in \mathbb{Z} \tag{19}$$

or, after substituting in the definition of ω and doing some algebra,

$$\lambda_n = \alpha + k\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots \tag{20}$$

These are the eigenvalues for the eigenvalue problem (12). Note that we now only use the positive integers; $n = 0$ gives $\lambda = \alpha$, but we are only considering $\lambda > \alpha$ here, and $n < 0$ gives the same set of values as $n > 0$, so there is no need to include negative n.

The eigenfunctions are

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \tag{21}$$

(I could include an arbitrary–but nonzero–constant in front of each eigenfunction, but it is not really necessary at this point. We *know* that any nonzero multiple of an eigenfunction is still an eigenfunction; moreover, we will soon introduce arbitrary constants when we write down the series solution to the PDE.)

At this point, we have solved the eigenfunction problem (12). Now, for each eigenvalue, we find the solution to the equation for h given in (11). This is simply

$$\begin{aligned} h_n(t) &= h_0 e^{-\lambda_n t} \\ &= h_0 e^{-(\alpha + k(n\pi/L)^2)t} \end{aligned} \tag{22}$$

where h_0 is an arbitrary constant. (This arbitrary constant is not important; in effect, it will be absorbed in the constant B_n to be introduced in a moment.)

Each function of the form $\phi_n(x)h_n(t)$ solves the original PDE and the boundary conditions (but not the initial conditions $u(x,0) = f(x)$. So we write the solution $u(x,t)$ as an infinite series

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} B_n e^{-(\alpha + k(n\pi/L)^2)t} \sin\left(\frac{n\pi x}{L}\right) \\ &= e^{-\alpha t} \sum_{n=1}^{\infty} B_n e^{-k(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right) \end{aligned} \tag{23}$$

(Compare this to equation 2.3.30 in the text, which is the solution to the case where $\alpha = 0$.)

To satisfy the initial condition, we must choose the coefficients B_n so that $u(x,0) = f(x)$. That is,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), \tag{24}$$

which tells us that the coefficients B_n are the Fourier sine series coefficients of the function $f(x)$:

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \tag{25}$$

Equation (23) with (25) is the solution to the original problem.

From (23), we see that $u(x,t)$ approaches zero as $t \to \infty$. The solution converges to the equilibrium solution that we found in part (a).

2.3.9 This is the same problem as Exercise 2.3.8, except that we assume $\alpha < 0$.

(a) The equation for an equilibrium solution is still (3), but since $\alpha < 0$, the general solution is

$$u_e(x) = c_1 \cos\left(\sqrt{\frac{-\alpha}{k}}\, x\right) + c_2 \sin\left(\sqrt{\frac{-\alpha}{k}}\, x\right) \tag{26}$$

To satisfy the boundary condition $u_e(0) = 0$, we must have $c_1 = 0$. At $x = L$, we must have

$$c_2 \sin\left(\sqrt{\frac{-\alpha}{k}}\, L\right) = 0 \tag{27}$$

If $\sqrt{-\alpha/k}\, L \neq n\pi$ for any integer n, then c_2 must be zero, and the only equilibrium solution is $u_e(x) = 0$. If, however, $\sqrt{-\alpha/k}L = n\pi$ for some integer n, then there is an equilibrium solution

$$u_e(x) = c_2 \sin\left(\sqrt{\frac{-\alpha}{k}}\, x\right) \tag{28}$$

wher c_2 is an arbitrary constant.

Before moving on to part (b), let's be sure we understand this result. Keep in mind that k and α are *physical parameters*; in a real heat equation problem, we can't "choose" these. They are determined by the physical material in the problem. What the above calculation says is that, for a given k, α and L, we would *normally* expect that *the only equilibrium solution is* $u_e(x) = 0$. *If* it turns out that $\sqrt{-\alpha/k}\, L$ happens to be an integer multiple of π, then there are nontrivial equilibrium solutions of the form given above. This does *not* say that, with k, α and L *fixed*, there is an equilibrium solution for any n.

(b) The formula for the solution to the problem when $\alpha < 0$ is the same as when $\alpha > 0$:

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-(\alpha + k(n\pi/L)^2)t} \sin\left(\frac{n\pi x}{L}\right) \tag{29}$$

where

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx \tag{30}$$

Now, consider the behavior of this solution as $t \to \infty$. The only dependence on t comes from the factors $\exp(-(\alpha + k(n\pi/L)^2)t)$.

If $-\alpha < k(\pi/L)^2$ (i.e. the coefficient of t in the exponent is negative when $n = 1$), then *all* the exponential factors will decay, and the solution will approach 0 as $t \to \infty$.

If $-\alpha > k(\pi/L)^2$, then the coefficient of t in $\exp(-(\alpha + k(\pi/L)^2)t)$ is positive, and the first term in the series solution will *grow* exponentially. (Well, it will if $B_1 \neq 0$. For an arbitrary $f(x)$, we expect that all the coefficients B_n will be nonzero.)

If $-\alpha = k(\pi/L)^2$, then the coefficient of t in the first exponential factor is zero, and the first term in the series solution is $B_1 \sin\left(\frac{n\pi x}{L}\right)$. The coefficients of t in the rest of the (infinitely many) exponential factors will be negative, so as $t \to \infty$,

$$u(x,t) \to B_1 \sin\left(\frac{n\pi x}{L}\right) \tag{31}$$

(This is, of course, a very special–and unlikely–case!)

So what about the case where $-\alpha = k(n\pi/L)^2$ for some $n > 1$? We know from part (a) that there is a nontrivial equilibrium in this special case, and if you look again at the series solution for $u(x,t)$ given above, you will see that in this case, the coefficient of t in the exponential factor of the n^{th} term will be zero. So one term in the series solution will be $B_n \sin(n\pi x/L)$; this particular term will not grow or decay. (This is precisely the equilibrium solution that we found in part (a).) However, the exponentials preceding this term in the series will have positive coefficients of t, so they will cause the solution to grow exponentially; the solution will *not* approach the equilibrium as $t \to \infty$. (We say that the equilibrium is *unstable*.) In the *very* special case that $B_i = 0$ for $1 \leq i < n$, then the solution would approach the equilibrium $B_n \sin(n\pi x/L)$.