Math 311 Applied Mathematics - Physical Sciences **Spring 2007**

Homework 3 Selected Solutions

3.5.4

$$\cosh x \sim \sum_{n=1}^{\infty} b_n \sin(n\pi x/L) \tag{1}$$

(a) To differentiate this sine series, we must use equation (3.4.13), with $f(x) = \cosh x$:

$$\sinh x \sim \frac{1}{L}(\cosh(L)-1) + \sum_{n=1}^{\infty}\left[\frac{n\pi}{L}b_n + \frac{2}{L}((-1)^n\cosh(L)-1)\right]\cos(n\pi x/L) \tag{2}$$

This is a cosine series, so to differentiate again, we can simply differentiate term-by-term:

$$\cosh x \sim \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)\left[\frac{n\pi}{L}b_n + \frac{2}{L}((-1)^n\cosh(L)-1)\right]\sin(n\pi x/L) \tag{3}$$

Now equate the coefficients in (1) and (3):

$$b_n = \left(\frac{n\pi}{L}\right)\left[\frac{n\pi}{L}b_n + \frac{2}{L}((-1)^n\cosh(L)-1)\right] \tag{4}$$

Then solve for b_n:

$$b_n = \frac{2n\pi\,(1-(-1)^n\cosh(L))}{L^2+(n\pi)^2} \tag{5}$$

(You can verify this answer by using the formula for the Fourier sine series coefficients; you'll have to integrate by parts twice.)

(b) We use definite integrals from 0 to x. In particular, $\int_0^x \cosh t\,dt = \sinh x - \sinh 0 = \sinh x$, and $\int_0^x \sinh t\,dt = \cosh x - \cosh 0 = \cosh x - 1$.

Integrating (1) results in

$$\begin{aligned}\sinh x &= \sum_{n=1}^{\infty} -\frac{L}{n\pi}b_n\,(\cos(n\pi x/L)-1)\\ &= \sum_{n=1}^{\infty}\frac{L}{n\pi}b_n + \sum_{n=1}^{\infty}\left(-\frac{L}{n\pi}b_n\right)\cos(n\pi x/L)\end{aligned} \tag{6}$$

The first infinite series on the right is the constant term of the Fourier cosine series for $\sinh x$, so we can replace it with $(1/L)\int_0^L \sinh(x)\,dx = (\cosh(L)-1)/L$:

$$\sinh x = \frac{1}{L}(\cosh(L)-1) + \sum_{n=1}^{\infty}\left[-\frac{L}{n\pi}b_n\right]\cos(n\pi x/L) \tag{7}$$

Integrate again from 0 to x:

$$\cosh(x) - 1 = \frac{1}{L}(\cosh(L) - 1)x + \sum_{n=1}^{\infty} \left[-\left(\frac{L}{n\pi}\right)^2 b_n \right] \sin(n\pi x/L) \tag{8}$$

or

$$\cosh(x) = 1 + \frac{1}{L}(\cosh(L) - 1)x + \sum_{n=1}^{\infty} \left[-\left(\frac{L}{n\pi}\right)^2 b_n \right] \sin(n\pi x/L) \tag{9}$$

We want the right side to be a Fourier sine series, so there can not be a constant term, nor a term with a constant multiplied by x. To fix this, we use the Fourier sine series for the constant 1 and for x:

$$1 \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - (-1)^n) \sin(n\pi x/L), \qquad x \sim \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \sin(n\pi x/L) \tag{10}$$

After substituting these series into (9) and collecting the coefficients of $\sin(n\pi x/L)$, we find

$$\begin{aligned} \cosh(x) &= \sum_{n=1}^{\infty} \left[\frac{2}{n\pi} \left(1 + \cosh(L)(-1)^{n+1}\right) - \left(\frac{L}{n\pi}\right)^2 b_n \right] \sin(n\pi x/L) \\ &= \sum_{n=1}^{\infty} \left[\frac{2}{n\pi} (1 - \cosh(L)(-1)^n) - \left(\frac{L}{n\pi}\right)^2 b_n \right] \sin(n\pi x/L) \end{aligned} \tag{11}$$

By equation the coefficients in (1) and (11), and then solving for b_n, we obtain (5).