Math 311 Applied Mathematics - Physical Sciences **Spring 2007**

Homework 4 Selected Solutions

1.5.9

(a) Since the temperatures on the inner and outer boundaries are constant (in particular, independent of θ), the equilibrium solution u_e will be independent of θ. The equilibrium solution will satisfy

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du_e}{dr}\right) = 0 \tag{1}$$

By integrating, we find

$$u_e(r) = c_1 + c_2 \ln r \tag{2}$$

We must have $u_e(r_1) = T_1$, and $u_e(r_2) = T_2$. A little algebra leads to

$$c_1 = \frac{T_1 \ln r_1 - T_2 \ln r_2}{\ln r_1 - \ln r_2}, \quad c_2 = \frac{T_2 - T_1}{\ln r_1 - \ln r_1} \tag{3}$$

After putting these values into the equation for $u_e(r)$ and doing a bit more algebra, we arrive at the solution in the text.

(b) Short answer only: $u_e(r) = T_1$.

1.5.14 An isobar is the same thing as a contour line of u.

Let $\vec{n}$ be a unit vector that is normal to the boundary, and let $\vec{\phi}$ be the flux. The condition for an insulated boundary is

$$\vec{\phi} \cdot \vec{n} = 0 \tag{4}$$

on the boundary. Fourier's Law says $\vec{\phi} = -K_0 \nabla u$, so the insulated boundary condition can be written

$$\nabla u \cdot \vec{n} = 0. \tag{5}$$

In other words, the gradient of u must be perpendicular to $\vec{n}$; this means the gradient is parallel to the boundary. Since the gradient is normal to the contour lines of u, the contour line must be perpendicular to the boundary.

2.5.1(b) The usual separation procedure $u(x,y) = h(x)\phi(y)$ leads the eigenvalue problem for ϕ:

$$\frac{d^2\phi}{dy^2} = -\lambda\phi, \quad \phi(0) = 0, \quad \phi(H) = 0. \tag{6}$$

We have solved this before; the eigenvalues are $\lambda = (n\pi/H)^2$, ($n = 1,2,3,\ldots$), and the corresponding eigenfunctions are $\phi(y) = \sin(n\pi y/H)$.

The differential equation for $h(x)$ is

$$\frac{d^2h}{dx^2} - \lambda h = 0, \tag{7}$$

where λ is an eigenvalue. We know $\lambda = (n\pi/H)^2 > 0$, so we could write the general solution to the differential equation as

$$h(x) = c_1 e^{n\pi x/H} + c_2 e^{-n\pi x/H}, \tag{8}$$

or as

$$h(x) = c_1 \cosh(n\pi x/H) + c_2 \sinh(n\pi x/H) \tag{9}$$

but because we will want $h(x)$ to satisfy a homogeneous boundary condition at $x = L$, it is more convenient to use the form

$$h(x) = c_1 \cosh(n\pi(x-L)/H) + c_2 \sinh(n\pi(x-L)/H). \tag{10}$$

From the original boundary condition for u given at $x = L$, we find $\frac{dh}{dx}(L) = 0$. We calculate

$$\frac{dh}{dx} = (c_1 n\pi/H)\sinh(n\pi(x-L)/H) + (c_2 n\pi/H)\cosh(n\pi(x-L)/H) \tag{11}$$

so at $x = L$, we must have $c_2 n\pi/H = 0$, which implies $c_2 = 0$. The solution to the differential equation for h that also satisfies the homogeneous boundary condition at $x = L$ is therefore

$$h(x) = c_1 \cosh(n\pi(x-L)/H) \tag{12}$$

We combine this with the eigenfunctions to obtain the solution as the series

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin(n\pi y/H)\cosh(n\pi(x-L)/H) \tag{13}$$

Now we must determine the coefficients b_n so that the nonhomogeneous boundary condition at $x = 0$ is satisfied. We find

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} (b_n n\pi/H)\sin(n\pi y/H)\sinh(n\pi(x-L)/H) \tag{14}$$

so

$$\frac{\partial u}{\partial x}(0,y) = -\sum_{n=1}^{\infty} (b_n n\pi/H)\sin(n\pi y/H)\sinh(n\pi L/H) = g(y) \tag{15}$$

(I have used $\sinh(-p) = -\sinh(p)$.) This says that the series is a Fourier sine series for $g(y)$, and therefore

$$-b_n n\pi \sinh(n\pi L/H)/H = \frac{2}{L}\int_0^L g(y)\sin(n\pi y/H)\,dy \tag{16}$$

or

$$b_n = \frac{-2H}{n\pi L \sinh(n\pi L/H)}\int_0^L g(y)\sin(n\pi y/H)\,dy \tag{17}$$

2.5.6(b) Laplace's equation in polar coordinates is equation (2.5.30) in the text. Separation of variables in the form $u(r,\theta) = G(r)\phi(\theta)$ leads to equation (2.5.36), and G and ϕ satisfy the homogeneous boundary conditions

$$\begin{aligned} &\frac{d\phi}{d\theta}(0) = 0 \quad (0 < r < a) \\ &\frac{d\phi}{d\theta}(\pi) = 0 \quad (0 < r < a) \\ &|G(0)| < \infty \end{aligned} \tag{18}$$

This is a "known" eigenvalue problem for ϕ, with $L = \pi$. The eigenvalues are $\lambda = n^2$ for $n = 0, 1, 2, \ldots$. The eigenfunction for $\lambda = 0$ is the constant function, and for $\lambda = n^2 > 0$, the eigenfunction is $\cos(n\theta)$.

For $\lambda = 0$, the equation for G has the solution given in equation (2.5.44), and by the same reasoning as in the text, we eliminate the function $\ln r$.

For $\lambda = n > 0$, the solution to the G equation is given by equation (2.5.43), and again the boundedness requirement at $r = 0$ allows us to eliminate the function r^{-n}.

For each eigenvalue, we now have a product $G(r)\phi(\theta)$ that solves the PDE and the homogeneous boundary conditions. We form the infinite series of these solutions:

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos(n\theta) \tag{19}$$

Finally, we must choose the coefficients so that the function satisfies the nonhomogeneous boundary condition at $r = a$. That is, we want

$$u(a,\theta) = A_0 + \sum_{n=1}^{\infty} A_n a^n \cos(n\theta) = g(\theta) \tag{20}$$

This says we want the series to be the Fourier cosine series for $g(\theta)$, which tells us that the coefficients must be

$$A_0 = \frac{1}{\pi}\int_0^{\pi} g(\theta)\, d\theta, \quad A_n = \frac{2}{\pi a^n}\int_0^{\pi} g(\theta)\cos(n\theta)\, d\theta \tag{21}$$

8.2.1(a) The differential equation is the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{22}$$

with the boundary conditions

$$u(0,t) = A, \quad \frac{\partial u}{\partial x}(L,t) = B \tag{23}$$

and the initial condition

$$u(x,0) = f(x) \tag{24}$$

For an equilibrium solution $u_e(x)$, we have

$$\frac{d^2 u_e}{dx^2} = 0, \quad \text{so} \quad u_e(x) = c_1 x + c_2 \tag{25}$$

and we can satisfy the boundary conditions with $c_1 = B$ and $c_2 = A$. Thus the equilibrium solution is

$$u_e(x) = Bx + A. \tag{26}$$

Now we write

$$u(x,t) = u_e(x) + w(x,t) = Bx + A + w(x,t) \tag{27}$$

Since $u(x,t)$ solves the original problem, we have

$$\frac{\partial(u_e(x) + w(x,t))}{\partial t} = \frac{\partial^2(u_e(x) + w(x,t))}{\partial x^2} \tag{28}$$

which simplies to

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} \tag{29}$$

since $\frac{\partial u_e}{\partial t} = 0$ and $\frac{d^2 u_e}{dx^2} = 0$. At $x = 0$ we have

$$A = u(0,t) = u_e(0) + w(0,t) = A + w(0,t), \tag{30}$$

so

$$w(0,t) = 0. \tag{31}$$

At $x = L$ we have

$$B = \frac{\partial u}{\partial x}(L,t) = \frac{\partial(u_e + w)}{\partial x}(L,t) = \frac{du_e}{dx}(L) + \frac{\partial w}{\partial x}(L,t) = B + \frac{\partial w}{\partial x}(L,t), \tag{32}$$

so

$$\frac{\partial w}{\partial x}(L,t) = 0. \tag{33}$$

At $t = 0$, we have

$$f(x) = u(x,0) = u_e(x) + w(x,0), \tag{34}$$

so

$$w(x,0) = f(x) - u_e(x) \tag{35}$$

To summarize, $w(x,t)$ satsfies the PDE (29), the homogeneous boundary conditions (31) and (33), and the initial condition (35)

We solve for $w(x,t)$ with the usual steps of separating variables, solving the appropriate eigenvalue problem, and forming a series solution. The solution is

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-((n-1/2)\pi/L)^2 kt} \sin((n-1/2)\pi x/L) \tag{36}$$

where

$$c_n = \frac{2}{L}\int_0^L (f(x) - u_e(x)) \sin((n-1/2)\pi x/L)\, dx \tag{37}$$

The solution to the original problem is then

$$u(x,t) = u_e(x) + w(x,t) \tag{38}$$

As $t \to \infty$, we see that $w(x,t) \to 0$, so $u(x,t) \to u_e(x)$.