

4.4.3

- (a) β is the damping (or friction) coefficient. If $\frac{\partial u}{\partial t} > 0$, then the string should decelerate, so $\frac{\partial^2 u}{\partial t^2}$ should be less than zero. This is the case if $\beta > 0$.

- (b) Assume $u(x,t) = \phi(x)h(t)$. Then the PDE becomes

$$\rho_0 \phi \frac{d^2 h}{dt^2} = T_0 \frac{d^2 \phi}{dx^2} h - \beta \phi \frac{dh}{dt}$$

$$\frac{\rho_0}{h} \frac{d^2 h}{dt^2} = \frac{T_0}{\phi} \frac{d^2 \phi}{dx^2} - \frac{\beta}{h} \frac{dh}{dt} \Rightarrow \frac{1}{h} \left(\rho_0 \frac{d^2 h}{dt^2} + \beta \frac{dh}{dt} \right) = \frac{T_0}{\phi} \frac{d^2 \phi}{dx^2} = -\lambda$$

$$\Rightarrow \rho_0 \frac{d^2 h}{dt^2} + \beta \frac{dh}{dt} + \lambda h = 0$$

$$\text{and } T_0 \frac{d^2 \phi}{dx^2} = -\lambda \phi, \quad \phi(0) = 0, \quad \phi(L) = 0$$

This is an eigenvalue problem; the solution is

$$\lambda_n = T_0 \left(\frac{n\pi}{L} \right)^2 \quad n = 1, 2, 3, \dots$$

$$\phi_n = \sin \left(\sqrt{\frac{\lambda_n}{T_0}} x \right) = \sin \left(\frac{n\pi x}{L} \right)$$

Put λ_n into the equation for h :

$$\rho_0 \frac{d^2 h}{dt^2} + \beta \frac{dh}{dt} + T_0 \left(\frac{n\pi}{L} \right)^2 h = 0$$

$$r = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 T_0 \left(\frac{n\pi}{L} \right)^2}}{2\rho_0} \quad \leftarrow \text{complex, since } \beta^2 < 4\rho_0 T_0 \left(\frac{n\pi}{L} \right)^2$$

$$\Rightarrow h(t) = c_1 e^{-\frac{\beta}{2\rho_0} t} \cos(w_n t) + c_2 e^{-\frac{\beta}{2\rho_0} t} \sin(w_n t) \quad \text{where } w_n = \sqrt{4\rho_0 T_0 \left(\frac{n\pi}{L} \right)^2 - \beta^2}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} e^{-\frac{\beta}{2\rho_0} t} [a_n \cos(w_n t) + b_n \sin(w_n t)] \sin \left(\frac{n\pi x}{L} \right)$$

4.4.3 continued

At $t = 0$,

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \Rightarrow a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\frac{\partial u}{\partial t}(x, 0) = \sum_{n=1}^{\infty} \left(b_n w_n - \frac{\beta}{2\rho_0} a_n \right) \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

$$\Rightarrow b_n w_n - \frac{\beta}{2\rho_0} a_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\Rightarrow b_n = \frac{1}{w_n} \left[\frac{\beta}{2\rho_0} a_n + \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx \right]$$

4.4.9

Equation (4.4.1) is the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Define the total energy

$$E = \int_0^L \frac{1}{2} \left(\frac{\partial u}{\partial t} \right)^2 dx + \int_0^L \frac{c^2}{2} \left(\frac{\partial u}{\partial x} \right)^2 dx$$

Differentiate with respect to t :

$$\begin{aligned} \frac{dE}{dt} &= \int_0^L \left(\frac{\partial u}{\partial t} \right) \left(\frac{\partial^2 u}{\partial t^2} \right) dx + \int_0^L c^2 \left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial^2 u}{\partial t \partial x} \right) dx \\ &\quad \downarrow \text{(4.4.1)} \qquad \qquad \qquad \downarrow \text{Integration by parts} \\ &= \int_0^L \left(\frac{\partial u}{\partial t} \right) \left(c^2 \frac{\partial^2 u}{\partial x^2} \right) dx + \underbrace{c^2 \left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial u}{\partial t} \right) \Big|_0^L}_{-} - \int_0^L c^2 \left(\frac{\partial^2 u}{\partial x^2} \right) \left(\frac{\partial u}{\partial t} \right) dx \\ &= c^2 \left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial u}{\partial t} \right) \Big|_0^L \end{aligned}$$

5.3.5

The eigenvalue problem is

$$\frac{d^2\phi}{dx^2} + \lambda \phi = 0 \quad \frac{d\phi}{dx}(0) = 0, \quad \frac{d\phi}{dx}(L) = 0$$

The eigenvalues are $\lambda_m = \left(\frac{m\pi}{L}\right)^2$ $m=0, 1, 2, \dots$

with eigenfunctions $\phi_m = \cos\left(\frac{m\pi x}{L}\right)$ (Note $m=0$ gives $\phi_0 = 1$)

(a) Yes: $\lambda_m = \left(\frac{m\pi}{L}\right)^2$; the smallest is $\lambda_0 = 0$.

(b) The $n+1$ eigenfunction is $\phi_{n+1} = \cos\left(\frac{(n+1)\pi x}{L}\right)$; it has $n+1$ zeros.



etc.

(c) Completeness: $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$ This is the Fourier cosine series representation of f . (section 3.3.2). Orthogonality is shown in section 2.4 (equation (2.4.22)).

(d) Rayleigh quotient ($p(x)=1$, $q(x)=0$, $\sigma(x)=1$)

$$\lambda = \frac{-p\phi \frac{d\phi}{dx} \Big|_0^L + \int_0^L [p \left(\frac{d\phi}{dx}\right)^2 - q\phi^2] dx}{\int_0^L \phi^2 dx} = \frac{\int_0^L \left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^L \phi^2 dx} \geq 0$$

From the Rayleigh quotient we see that there are no negative eigenvalues.

5.3.7

The given eigenvalue problem has periodic B.C.s

$$\text{The eigenvalues are } \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n=0, 1, 2, \dots$$

The eigenfunction for λ_0 is $\phi_0 = 1$

For $n \geq 1$, the eigenvalue λ_n has two eigenfunctions, $\sin\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{n\pi x}{L}\right)$.

Now check the statements of the theorem:

1. Valid
2. Valid
3. Not valid; for $n \geq 1$, λ_n has two eigenfunctions.
4. Valid. (This is just the Fourier series.)
5. Valid.

5.3.9

(a) Multiply by $\frac{1}{x}$:

$$x \frac{d^2\phi}{dx^2} + \frac{d\phi}{dx} + \frac{\lambda}{x}\phi = 0$$

$$\underbrace{\frac{d}{dx} \left(x \frac{d\phi}{dx} \right) + \lambda \left(\frac{1}{x} \right) \phi}_{= 0} = 0$$

this is the Sturm-Liouville form:

$$p(x) = x, \quad q(x) = 0, \quad \sigma(x) = \frac{1}{x}$$

Since the domain is $1 \leq x \leq b$, this is a regular Sturm-Liouville problem.

(b) In this case, the Rayleigh quotient is

$$\lambda = \frac{\int_1^b x \left(\frac{d\phi}{dx} \right)^2 dx}{\int_1^b \phi^2 x dx} \geq 0$$

$$(c) \quad x^2 \frac{d^2\phi}{dx^2} + x \frac{d\phi}{dx} + \lambda \phi = 0 \quad \phi(1) = 0, \quad \phi(b) = 0$$

Try $\phi = x^p$; so $\frac{d\phi}{dx} = px^{p-1}$ and $\frac{d^2\phi}{dx^2} = p(p-1)x^{p-2}$

$$\Rightarrow p(p-1)x^p + px^p + \lambda x^p = 0$$

$$\Rightarrow p^2 + \lambda = 0 \quad p = \pm \sqrt{-\lambda}$$

If $\lambda > 0$, p is complex: $p = \pm \sqrt{\lambda}i$

$$\text{then } x^{\sqrt{\lambda}i} = (e^{\ln x})^{\sqrt{\lambda}i} = e^{(\sqrt{\lambda} \ln x)i} = \cos(\sqrt{\lambda} \ln x) + i \sin(\sqrt{\lambda} \ln x)$$

$$\text{and } x^{-\sqrt{\lambda}i} = \dots = \cos(\sqrt{\lambda} \ln x) - i \sin(\sqrt{\lambda} \ln x)$$

We can use the real and imaginary parts of these solutions to form the real-valued general solution

$$\phi(x) = c_1 \cos(\sqrt{\lambda} \ln x) + c_2 \sin(\sqrt{\lambda} \ln x)$$

Now try to satisfy the boundary conditions:

$$\phi(1) = c_1 = 0$$

$$\phi(b) = c_2 \sin(\sqrt{\lambda} \ln b) = 0$$

$$\Rightarrow \sqrt{\lambda} \ln b = n\pi$$

$$\lambda = \left(\frac{n\pi}{\ln b}\right)^2$$

so we have eigenvalues $\lambda_n = \left(\frac{n\pi}{\ln b}\right)^2$ $n=1, 2, 3, \dots$ (Infinite number, no largest)

with eigenfunctions $\phi_n(x) = \sin(n\pi \frac{\ln x}{\ln b})$

check $\lambda=0$:

$$x^2 \frac{d^2\phi}{dx^2} + x \frac{d\phi}{dx} = 0 \quad \phi(1)=0, \quad \phi(b)=0$$

$$\Rightarrow \frac{d}{dx} \left(x \frac{d\phi}{dx} \right) = 0 \Rightarrow x \frac{d\phi}{dx} = c_2 \Rightarrow \phi = c_1 + c_2 \ln x$$

check B.C.s: $\phi(1) = c_1 = 0$

$$\phi(b) = c_2 \ln b = 0 \quad \text{but } b > 1, \text{ so } \ln b \neq 0 \Rightarrow c_2 = 0$$

$\lambda=0$ is not an eigenvalue.

(d) The weight function is $w(x) = \frac{1}{x}$.

$$\int_1^b \sin\left(n\pi \frac{\ln x}{\ln b}\right) \sin\left(m\pi \frac{\ln x}{\ln b}\right) \frac{1}{x} dx$$

Let $u = \ln x$, $du = \frac{1}{x} dx$

$$= \int_0^{\ln b} \sin\left(n\pi u\right) \sin\left(m\pi u\right) du = \begin{cases} 0 & m \neq n \\ \frac{\ln b}{2} & m = n \end{cases}$$

This is equation (2.3.23), with $L = \ln b$.

(e) The n^{th} eigenfunction is $\phi_n(x) = \sin\left(n\pi \frac{\ln x}{\ln b}\right)$

Since $1 \leq x \leq b$, we have $0 \leq \frac{\ln x}{\ln b} \leq 1$

Let $\theta = \frac{\ln x}{\ln b}$; $\sin(n\pi\theta)$ has $n-1$ zeros in the (open)

interval $0 < \theta < 1$, so $\sin(n\pi \frac{\ln x}{\ln b})$ has $n-1$ zeros

in the interval $1 < x < b$.

5.4.1

$$cp \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(K_0 \frac{\partial u}{\partial x} \right) + \alpha u$$

Assume $\alpha < 0$ (note that $cp > 0$ and $K_0 > 0$)

(a) Separation of variables $u(x, t) = \phi(x) h(t)$ gives

$$\frac{dh}{dt} = -\lambda h \quad (5.2.6)$$

and $\frac{d}{dx} \left(K_0 \frac{d\phi}{dx} \right) + \alpha \phi + \lambda cp \phi = 0 \quad \phi(0) = 0, \phi(L) = 0 \quad (5.2.7)$

This is a regular Sturm-Liouville problem, with $p(x) = K_0$, $q(x) = \alpha$, $\sigma(\omega) = cp$.

The Rayleigh quotient is

$$\lambda = \frac{\int_0^L \left[K_0 \left(\frac{d\phi}{dx} \right)^2 - \alpha \phi^2 \right] dx}{\int_0^L \phi^2 cp dx} > 0 \quad (\text{since } \alpha < 0)$$

($\lambda = 0$ is not possible, since that would imply $\phi \equiv 0$.)

(b) Let λ_n be the set of eigenvalues, with eigenfunctions $\phi_n(x)$, ($n=1, 2, \dots$)

Then

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \phi_n(x)$$

At $t = 0$,

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \phi_n(x) = f(x)$$

so

$$a_n = \frac{\int_0^L f(x) \phi_n(x) cp dx}{\int_0^L \phi_n^2(x) cp dx}$$

(c) Since $\lambda_n > 0$, $e^{-\lambda_n t} \rightarrow 0$ as $t \rightarrow \infty$, and so

$$\lim_{t \rightarrow \infty} u(x, t) = 0$$

5.5.1

(a) u and v satisfy $\phi(0)=0$ and $\phi(L)=0$.

Then

$$\left. p\left(u \frac{dv}{dx} - v \frac{du}{dx}\right)\right|_0^L = p(L) \left(u(L) \overset{0}{\underset{\cancel{dx}}{\frac{dv}{dx}}} (L) - v(L) \overset{0}{\underset{\cancel{dx}}{\frac{du}{dx}}} (L) \right) - p(0) \left(u(0) \overset{0}{\underset{\cancel{dx}}{\frac{dv}{dx}}} (0) - v(0) \overset{0}{\underset{\cancel{dx}}{\frac{du}{dx}}} (0) \right)$$

$$= 0 \quad \checkmark$$

(c) u and v satisfy $\frac{d\phi(0)}{dx} = h\phi(0) = 0$ and $\frac{d\phi(L)}{dx} = 0$

Then

$$\left. p\left(u \frac{dv}{dx} - v \frac{du}{dx}\right)\right|_0^L = p(L) \left(u(L) \overset{0}{\underset{\cancel{dx}}{\frac{dv}{dx}}} (L) - v(L) \overset{0}{\underset{\cancel{dx}}{\frac{du}{dx}}} (L) \right) - p(0) \left(u(0) \overset{0}{\underset{\cancel{dx}}{\frac{dv}{dx}}} (0) - v(0) \overset{0}{\underset{\cancel{dx}}{\frac{du}{dx}}} (0) \right)$$

$$= -p(0) \left(u(0) \cancel{[h v(0)]} - v(0) \cancel{[h u(0)]} \right) = 0 \quad \checkmark$$

(d) u and v satisfy $\phi(a) = \phi(b)$ and $p(a) \frac{d\phi}{dx}(a) = p(b) \frac{d\phi}{dx}(b)$.

Then

$$\left. p\left(u \frac{dv}{dx} - v \frac{du}{dx}\right)\right|_a^b = p(b) \left(u(b) \overset{0}{\underset{\cancel{dx}}{\frac{dv}{dx}}} (b) - v(b) \overset{0}{\underset{\cancel{dx}}{\frac{du}{dx}}} (b) \right) - p(a) \left(u(a) \overset{0}{\underset{\cancel{dx}}{\frac{dv}{dx}}} (a) - v(a) \overset{0}{\underset{\cancel{dx}}{\frac{du}{dx}}} (a) \right)$$

$$= p(b) \cancel{u(b) \frac{dv}{dx}(b)} - p(b) \cancel{v(b) \frac{du}{dx}(b)} - p(b) \cancel{u(b) \frac{dv}{dx}(b)} + p(b) \cancel{v(b) \frac{du}{dx}(b)}$$

$$= 0 \quad \checkmark$$

(f) u and v satisfy $\phi(L)=0$ and $\phi(x)$ is bounded and $\lim_{x \rightarrow 0} p(x) \frac{d\phi}{dx} = 0$

Then

$$\left. p\left(u \frac{dv}{dx} - v \frac{du}{dx}\right)\right|_0^L = p(L) \left(u(L) \overset{0}{\underset{\cancel{dx}}{\frac{dv}{dx}}} (L) - v(L) \overset{0}{\underset{\cancel{dx}}{\frac{du}{dx}}} (L) \right) - \lim_{x \rightarrow 0} p(x) \left(u(x) \overset{0}{\underset{\cancel{dx}}{\frac{dv}{dx}}} (x) - v(x) \overset{0}{\underset{\cancel{dx}}{\frac{du}{dx}}} (x) \right)$$

$$= -u(x) \lim_{x \rightarrow 0} p(x) \overset{0}{\underset{\cancel{dx}}{\frac{dv}{dx}}} (x) - v(x) \lim_{x \rightarrow 0} p(x) \overset{0}{\underset{\cancel{dx}}{\frac{du}{dx}}} (x)$$

$$= 0 \quad \checkmark$$

5.5.8

$$L = \frac{d^4}{dx^4}$$

(a) $uL(v) - vL(u) = u \frac{d^4v}{dx^4} - v \frac{d^4u}{dx^4}$

$$= \frac{d}{dx} \left[\left(u \frac{d^3v}{dx^3} - v \frac{d^3u}{dx^3} \right) - \left(\frac{du}{dx} \frac{d^2v}{dx^2} - \frac{dv}{dx} \frac{d^2u}{dx^2} \right) \right]$$

(b) $\int_0^1 [uL(v) - vL(u)] dx = \left[\left(u \frac{d^3v}{dx^3} - v \frac{d^3u}{dx^3} \right) - \left(\frac{du}{dx} \frac{d^2v}{dx^2} - \frac{dv}{dx} \frac{d^2u}{dx^2} \right) \right]_0^1$

(c) If u and v satisfy $\phi(0) = 0$, $\phi(1) = 0$, $\frac{d\phi}{dx}(0) = 0$, $\frac{d^2\phi}{dx^2}(1) = 0$ then

$$\begin{aligned} \int_0^1 [uL(v) - vL(u)] dx &= \left(u(1) \overset{0}{\cancel{\frac{d^3v}{dx^3}}}(1) - v(1) \overset{0}{\cancel{\frac{d^3u}{dx^3}}}(1) \right) - \left(\frac{du}{dx}(1) \overset{0}{\cancel{\frac{d^2v}{dx^2}}}(1) - \frac{dv}{dx}(1) \overset{0}{\cancel{\frac{d^2u}{dx^2}}}(1) \right) \\ &\quad - \left(u(0) \overset{0}{\cancel{\frac{d^3v}{dx^3}}}(0) - v(0) \overset{0}{\cancel{\frac{d^3u}{dx^3}}}(0) \right) + \left(\frac{du}{dx}(0) \overset{0}{\cancel{\frac{d^2v}{dx^2}}}(0) - \frac{dv}{dx}(0) \overset{0}{\cancel{\frac{d^2u}{dx^2}}}(0) \right) \\ &= 0 \quad \checkmark \end{aligned}$$

(d) For example, $\frac{d\phi}{dx}(0) = 0$, $\frac{d^3\phi}{dx^3}(0) = 0$, $\phi(1) = 0$, $\frac{d\phi}{dx}(1) = 0$

(e) Suppose ϕ_m and ϕ_n are eigenfunctions associated with the eigenvalues λ_m and λ_n , and $\lambda_m \neq \lambda_n$. From part (c) we know $\int_0^1 \phi_m L(\phi_n) - \phi_n L(\phi_m) dx = 0$.

Then

$$\begin{aligned} 0 &= \int_0^1 \phi_m L(\phi_n) - \phi_n L(\phi_m) dx = \int_0^1 \phi_m (-\lambda_n e^x \phi_n) - \phi_n (-\lambda_m e^x \phi_m) dx \\ &= (\lambda_m - \lambda_n) \int_0^1 \phi_m \phi_n e^x dx \end{aligned}$$

Since $\lambda_m \neq \lambda_n$, we must have

$$\int_0^1 \phi_m \phi_n e^x dx = 0 \quad (\text{The weighting function is } e^x.)$$

5.5.9

The differential equation is

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0$$

Multiply by ϕ and integrate:

$$\int_0^1 \phi \frac{d^4\phi}{dx^4} dx + \lambda \int_0^1 \phi^2 e^x dx = 0$$

so

$$\lambda = \frac{- \int_0^1 \phi \frac{d^4\phi}{dx^4} dx}{\int_0^1 \phi^2 e^x dx}$$

Now

$$\int_0^1 \phi \frac{d^4\phi}{dx^4} dx = \cancel{\phi \frac{d^3\phi}{dx^3}} \Big|_0^1 - \int_0^1 \frac{d\phi}{dx} \frac{d^3\phi}{dx^3} dx \quad (\text{Integration by parts})$$

$$= - \cancel{\frac{d\phi}{dx} \frac{d^2\phi}{dx^2}} \Big|_0^1 + \int_0^1 \left(\frac{d^2\phi}{dx^2} \right)^2 dx \quad (\text{I.B.P. again})$$

Thus

$$\lambda = \frac{- \int_0^1 \left(\frac{d^2\phi}{dx^2} \right)^2 dx}{\int_0^1 \phi^2 e^x dx} \leq 0$$

$$\text{Check } \lambda = 0 : \frac{d^4\phi}{dx^4} = 0 \Rightarrow \phi = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

$$\phi(0) = 0 \Rightarrow c_0 = 0 \quad \frac{d\phi}{dx}(0) = 0 \Rightarrow c_1 = 0$$

$$\phi(1) = 0 \Rightarrow c_2 + c_3 = 0 \quad \frac{d^4\phi}{dx^4}(1) = 2c_2 + 6c_3 = 0 \Rightarrow -4c_2 = 0 \quad c_2 = 0 \\ c_3 = -c_2$$

so $\lambda = 0 \Rightarrow \phi = 0$; $\lambda = 0$ is not an eigenvalue.

5.6.1

$$(a) \quad \frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0 \quad \frac{d\phi}{dx}(0) = 0, \quad \phi(1) = 0$$

$$p = 1, \quad q = -x^2, \quad \sigma = 1$$

Rayleigh quotient:

$$\lambda_1 = \min \frac{\int_0^1 \left[\left(\frac{du}{dx} \right)^2 + x^2 u^2 \right] dx}{\int_0^1 u^2 dx}$$

$$\text{Try } u = \boxed{1 - x^2}$$

$$\frac{du}{dx} = -2x$$

$$u^2 = 1 - 2x^2 + x^4$$

$$\text{Numerator} \quad \int_0^1 \left(\frac{du}{dx} \right)^2 + x^2 u^2 dx = \int_0^1 4x^2 + x^2 - 2x^4 + x^6 dx = \frac{4}{3} + \frac{1}{3} - \frac{2}{5} + \frac{1}{7} = \frac{148}{105}$$

$$\text{Denom} \quad \int_0^1 u^2 dx = \int_0^1 1 - 2x^2 + x^4 dx = 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}$$

$$\lambda \leq \frac{\left(\frac{148}{105}\right)}{\left(\frac{8}{15}\right)} = \frac{37}{14} \approx 2.643$$

OR Try $\boxed{u = \cos\left(\frac{\pi x}{2}\right)}$. Then

$$\frac{\int_0^1 \left(\left(\frac{du}{dx} \right)^2 + x^2 u^2 \right) dx}{\int_0^1 u^2 dx} = \dots = \frac{3\pi^4 + 4\pi^2 - 24}{12\pi^2} \approx 2.598$$

$$\text{So } \lambda \leq 2.598$$