

5.7.2

With nonconstant thermal properties, the heat

equation is

$$cp \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( K_0 \frac{\partial u}{\partial x} \right)$$

(1.2.9, with  $Q=0$ )where  $c, p$  and  $K_0$  depend on  $x$ . The given B.C.s are

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

Separation of variables with  $u(x,t) = \phi(x)h(t)$  gives

$$cp\phi \frac{dh}{dt} - \frac{d}{dx} \left( K_0 \frac{d\phi}{dx} \right) h \Rightarrow \frac{1}{h} \frac{dh}{dt} = \frac{1}{cp\phi} \frac{d}{dx} \left( K_0 \frac{d\phi}{dx} \right) = -\lambda$$

The equation for  $\phi$  is a regular Sturm-Liouville problem:

$$\frac{d}{dx} \left( K_0 \frac{d\phi}{dx} \right) + \lambda cp\phi = 0, \quad \phi(0) = 0, \quad \phi(L) = 0$$

$$[p(x) = K_0(x), \quad q(x) = 0, \quad \sigma(x) = c(x)p(x)]$$

So

$$\lambda_1 = \min_u \frac{\int_0^L K_0(x) \left( \frac{du}{dx} \right)^2 dx}{\int_0^L u^2 cp dx}$$

We are told  $K_{\min} \leq K_0 \leq K_{\max}$ , and  $cp_{\min} \leq cp \leq cp_{\max}$ ;

then

$$\lambda_1 = \min_u \frac{\int_0^L K_0(x) \left( \frac{du}{dx} \right)^2 dx}{\int_0^L u^2 cp dx} \leq \min_u \frac{K_{\max} \int_0^L \left( \frac{du}{dx} \right)^2 dx}{cp_{\min} \int_0^L u^2 dx} = \frac{K_{\max}}{cp_{\min}} \left[ \min_u \frac{\int_0^L \left( \frac{du}{dx} \right)^2 dx}{\int_0^L u^2 dx} \right]$$

and

$$\lambda_1 = \min_u \frac{\int_0^L K_0(x) \left( \frac{du}{dx} \right)^2 dx}{\int_0^L u^2 cp dx} \geq \min_u \frac{K_{\min} \int_0^L \left( \frac{du}{dx} \right)^2 dx}{cp_{\max} \int_0^L u^2 dx} = \frac{K_{\min}}{cp_{\max}} \left[ \min_u \frac{\int_0^L \left( \frac{du}{dx} \right)^2 dx}{\int_0^L u^2 dx} \right]$$

Next we observe that  $\min_u \frac{\int_0^L (\frac{du}{dx})^2 dx}{\int_0^L u^2 dx}$  is the smallest eigenvalue of the Sturm-Liouville problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad \phi(0) = 0, \quad \phi(L) = 0$$

which is  $(\frac{\pi}{L})^2$ . Combining ④ and ⑤, we have

$$\frac{K_{\min}}{C P_{\max}} \left(\frac{\pi}{L}\right)^2 \leq \lambda_1 \leq \frac{K_{\max}}{C P_{\min}} \left(\frac{\pi}{L}\right)^2$$

Since the time-dependent part of the is  $e^{-\lambda_1 t}$ , these bounds on  $\lambda_1$  give upper and lower bounds on the slowest exponential rate of decay.

7.2.1  $\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$  (7.2.1)

Let  $u(x, y, t) = \phi(x, y) h(t)$

Then  $\phi \frac{d^2 h}{dt^2} = c^2 \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) h \Rightarrow \frac{1}{c^2 h} \frac{d^2 h}{dt^2} = \frac{1}{\phi} \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = -\lambda$

The equation for  $\phi$  is then

$$\underbrace{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}}_{\nabla^2 \phi} = -\lambda \phi \Rightarrow \nabla^2 \phi = -\lambda \phi \quad (7.2.14)$$

7.2.2 Equation (7.1.1) is

$$\frac{\partial u}{\partial t} = k \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = k \nabla^2 u$$

Let  $u(x, y, t) = \phi(x, y) h(t)$ . Then

$$\phi \frac{dh}{dt} = k \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) h = k(\nabla^2 \phi) h \Rightarrow \frac{1}{k h} \frac{dh}{dt} = \frac{1}{\phi} \nabla^2 \phi = -\lambda$$

so the equation for  $\phi$  is

$$\nabla^2 \phi = -\lambda \phi. \quad (7.2.14)$$

7.3.3 Separate variables  $u(x, y, t) = \phi(x, y) h(t)$

$$\Rightarrow \phi \frac{dh}{dt} = k_1 \frac{\partial^2 \phi}{\partial x^2} h + k_2 \frac{\partial^2 \phi}{\partial y^2} h = \left( k_1 \frac{\partial^2 \phi}{\partial x^2} + k_2 \frac{\partial^2 \phi}{\partial y^2} \right) h$$

$$\Rightarrow \frac{1}{h} \frac{dh}{dt} = \frac{1}{\phi} \left( k_1 \frac{\partial^2 \phi}{\partial x^2} + k_2 \frac{\partial^2 \phi}{\partial y^2} \right) = -\lambda$$

$$\Rightarrow \frac{dh}{dt} = -\lambda h \quad \text{and} \quad k_1 \frac{\partial^2 \phi}{\partial x^2} + k_2 \frac{\partial^2 \phi}{\partial y^2} = -\lambda \phi \quad (\phi \text{ has the same B.C.s as } u)$$

To solve for  $\phi$ , we again separate:  $\phi(x, y) = f(x)g(y)$

$$\Rightarrow k_1 \frac{d^2 f}{dx^2} g + k_2 f \frac{d^2 g}{dy^2} = -\lambda f g$$

$$\Rightarrow \frac{1}{f} \frac{d^2 f}{dx^2} = \frac{-1}{k_1 g} \left( \lambda g + k_2 \frac{d^2 g}{dy^2} \right) = -\mu$$

$$\Rightarrow \frac{d^2 f}{dx^2} = -\mu f \quad \text{with D.C.s } f(0) = 0, f(L) = 0$$

$$\text{and } \frac{d^2 g}{dy^2} + \left( \frac{\lambda - \mu k_1}{k_2} \right) g = 0 \quad \frac{dg}{dy}(0) = 0, \quad \frac{dg}{dy}(H) = 0$$

The solution to the equation for  $f$  is

$$\mu_n = \left(\frac{n\pi}{L}\right)^2 \quad f_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

The equation for  $g$  is then

$$\frac{d^2 g}{dy^2} + \left( \frac{\lambda - \left(\frac{n\pi}{L}\right)^2 k_1}{k_2} \right) g = 0 \quad \frac{dg}{dy}(0) = 0, \quad \frac{dg}{dy}(H) = 0$$

$$\Rightarrow \frac{\lambda_{0n} - \left(\frac{n\pi}{L}\right)^2 k_1}{k_2} = 0 \Rightarrow \lambda_{0n} = \left(\frac{n\pi}{L}\right)^2 k_1 \quad g_{0n}(y) = 1$$

$$\frac{\lambda_{mn} - \left(\frac{n\pi}{L}\right)^2 k_1}{k_2} = \left(\frac{m\pi}{H}\right)^2 \Rightarrow \lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 k_1 + \left(\frac{m\pi}{H}\right)^2 k_2 \quad g_{mn}(y) = \cos\left(\frac{m\pi y}{H}\right)$$

so the eigenvalues are

$$\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 k_1 + \left(\frac{m\pi}{H}\right)^2 k_2 \quad n=1, 2, 3, \dots$$

$m=0, 1, 2, \dots$

with eigenfunctions

$$q_{mn}(x, y) = \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right)$$

The solution to the equation for  $h$  is  $e^{-\lambda_{mn}t}$   
the general solution to the PDE is then

$$u(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} c_{mn} e^{-\left[\left(\frac{n\pi}{L}\right)^2 k_1 + \left(\frac{m\pi}{H}\right)^2 k_2\right]t} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right)$$

At  $t=0$ ,

$$u(x, y, 0) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} c_{mn} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right) = f(x, y)$$

Then

$$c_{mn} = \frac{\int_0^L \int_0^H f(x, y) \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right) dy dx}{\int_0^L \int_0^H \left[\sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right)\right]^2 dy dx}$$

$$= \frac{4}{LH} \int_0^L \int_0^H f(x, y) \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right) dy dx$$

7.3.5

(a) this is the damped wave equation in two dimensions.  
 $k$  is the damping (ie friction) coefficient.

(b) (Just the facts...)

$$\frac{d^2h}{dt^2} + k \frac{dh}{dt} + \lambda c^2 h = 0$$

$$\frac{dg}{dy^2} + \lambda g = 0$$

$$\frac{df}{dx^2} + (\lambda - \mu) f = 0$$

where  $\lambda$  and  $\mu$  are separation constants,  
 (other answers are possible.)

7.4.1 (Just the facts...)

$$(a) \lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 \quad n=1, 2, 3, \dots$$

$$m=0, 1, 2, \dots$$

$$(b) \text{ If } L=H, \text{ then } \lambda_{mn} = \left(\frac{\pi}{L}\right)^2 (n^2 + m^2) = \lambda_{nm}$$

and if  $m \neq n$ , the eigenvalue  $\lambda_{mn}$  has two eigenfunctions

$$\cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right) \quad \text{and} \quad \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)$$

(also assuming  $m > 0, n > 0$ )

$$(c) \phi_{mn} = \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right)$$

$$\iint_R \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right) \cos\left(\frac{\tilde{m}\pi x}{L}\right) \sin\left(\frac{\tilde{n}\pi y}{H}\right) dx dy$$

$$= \int_0^H \sin\left(\frac{n\pi y}{H}\right) \sin\left(\frac{\tilde{n}\pi y}{H}\right) \int_0^L \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{\tilde{m}\pi x}{L}\right) dx dy$$

$$= \underbrace{\left( \int_0^H \sin\left(\frac{n\pi y}{H}\right) \sin\left(\frac{\tilde{n}\pi y}{H}\right) dy \right)}_{=0 \text{ unless } n=\tilde{n}} \underbrace{\left( \int_0^L \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{\tilde{m}\pi x}{L}\right) dx \right)}_{=0 \text{ unless } m=\tilde{m}}$$

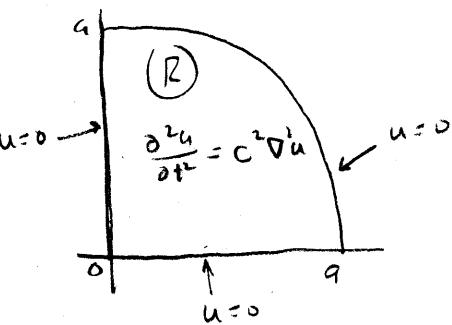
Thus

$$\iint_R \phi_{mn} \phi_{\tilde{m}\tilde{n}} dA = 0 \text{ unless } m=\tilde{m} \text{ and } n=\tilde{n}.$$

7.4.2 From equation (7.4.6),  $\lambda = \frac{-\oint \phi \nabla \phi \cdot \hat{n} dx + \iint_R |\nabla \phi|^2 dx dy}{\iint_R \phi^2 dx dy}$

$$= \frac{\iint_R |\nabla \phi|^2 dx dy}{\iint_R \phi^2 dx dy} \geq 0. \quad (\because \oint \phi \nabla \phi \cdot \hat{n} dx = 0 \text{ since } \phi=0 \text{ on the boundary.})$$

7.7.3



Let  $R$  be the quarter disk.

(a)

Separate variables, as in section 7.7.2. We obtain equations (7.7.11), (7.7.12) and (7.7.13). In this case, however, the boundary conditions for (7.7.12) are  $g(0)=0$  and  $g(\frac{\pi}{2})=0$

The eigenvalues for the  $g$  equation are  $\mu=(2m)^2$  ( $m=1, 2, 3, \dots$ ) with eigenfunctions  $g=\sin(2m\theta)$ .

Then (7.7.13) becomes

$$r \frac{d}{dr} \left( r \frac{df}{dr} \right) + (\lambda r^2 - (2m)^2) f = 0 \quad |f(0)| < \infty, \quad f(a) = 0$$

The change of variable  $z = \sqrt{\lambda} r$  leads to the Bessel equation

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - (2m)^2) f = 0$$

The solutions that are bounded at 0 are

$$f = J_{2m}(z) - J_{2m}(\sqrt{\lambda} a)$$

to satisfy  $J_{2m}(\sqrt{\lambda} a) = 0$ , we must have

$$\lambda_{mn} = \left( \frac{z_{(2m)n}}{a} \right)^2, \quad \text{and the eigenfunctions are } f_{mn}(r) = J_{2m}\left( \frac{z_{(2m)n}}{a} r \right)$$

Then (7.7.11) is

$$\frac{d^2 h}{dt^2} = -\lambda_{mn} c^2 h \Rightarrow h(t) = c_1 \cos(\sqrt{\lambda_{mn}} ct) + c_2 \sin(\sqrt{\lambda_{mn}} ct) \\ = c_1 \cos\left(\frac{z_{(2m)n}}{a} ct\right) + c_2 \sin\left(\frac{z_{(2m)n}}{a} ct\right)$$

so the frequencies of vibration are

$$\frac{z_{(2m)n}}{a} c$$

$$m = 1, 2, 3, \dots, \quad n = 1, 2, 3, \dots$$

(b) The general solution is

$$u(r, \theta, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} J_{2m}(\sqrt{\lambda_{mn}} r) \sin(2m\theta) [A_{mn} \cos(\sqrt{\lambda_{mn}} ct) + B_{mn} \sin(\sqrt{\lambda_{mn}} ct)]$$

since  $\frac{\partial u}{\partial t}(r, \theta, 0) = 0$ , we know  $B_{mn} = 0$ .

and since

$$u(r, \theta, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} J_{2m}(\sqrt{\lambda_{mn}} r) \sin(2m\theta) = g(r, \theta),$$

we have

$$A_{mn} = \frac{\iint_R g(r, \theta) J_{2m}(\sqrt{\lambda_{mn}} r) \sin(2m\theta) dA}{\iint_R [J_{2m}(\sqrt{\lambda_{mn}} r) \sin(2m\theta)]^2 dA}$$

$$= \frac{\int_0^{\pi/2} \int_0^a g(r, \theta) J_{2m}(\sqrt{\lambda_{mn}} r) \sin(2m\theta) r dr d\theta}{\int_0^{\pi/2} \int_0^a [J_{2m}(\sqrt{\lambda_{mn}} r) \sin(2m\theta)]^2 r dr d\theta}$$

7.7.7

The derivation follows the same steps as the wave equation, except that the equation for  $h(t)$  is

$$\frac{dh}{dt} = -\lambda k h \Rightarrow h = ce^{-\lambda kt}$$

The solutions to Helmholtz equation in the disk is

$$\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2, \quad \phi_{mn} = \begin{cases} J_m(z_{mn} \frac{r}{a}) \cos(m\theta) \\ J_m(z_{mn} \frac{r}{a}) \sin(m\theta) \end{cases} \quad \begin{matrix} m=1, 3, 5, \dots \\ n=1, 3, 5, \dots \end{matrix}$$

$$\text{and } \lambda_{0n} = \left(\frac{z_{0n}}{a}\right)^2, \quad \phi_{0n} = J_0(z_{0n} \frac{r}{a}) \quad n=1, 3, 5, \dots$$

We can write the general solution as

$$u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left[ A_{mn} J_m(z_{mn} \frac{r}{a}) \cos(m\theta) + B_{mn} J_m(z_{mn} \frac{r}{a}) \sin(m\theta) \right] e^{-k(z_{mn}/a)^2 t}$$

At  $t=0$ ,

$$u(r, \theta, 0) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} J_m(z_{mn} \frac{r}{a}) \cos(m\theta) + B_{mn} J_m(z_{mn} \frac{r}{a}) \sin(m\theta) = f(r, \theta)$$

$$A_{mn} = \frac{\int_{-\pi}^{\pi} \int_0^a f(r, \theta) J_m(z_{mn} \frac{r}{a}) \cos(m\theta) r dr d\theta}{\int_{-\pi}^{\pi} \int_0^a [J_m(z_{mn} \frac{r}{a}) \cos(m\theta)]^2 r dr d\theta}$$

$$B_{mn} = \frac{\int_{-\pi}^{\pi} \int_0^a f(r, \theta) J_m(z_{mn} \frac{r}{a}) \sin(m\theta) r dr d\theta}{\int_{-\pi}^{\pi} \int_0^a [J_m(z_{mn} \frac{r}{a}) \sin(m\theta)]^2 r dr d\theta}$$