Math 312 Lecture Notes
Competing Species and Nonlinear Phase Plane Analysis

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Competing Species

We consider an example that models the populations of two species that are competing for a common resource. In the absence of the other species, each species grows according to a logistic equation. However, the presence of one species lowers the *per capita* growth rate of the other species. One way to write the equations for this system is

$$
\begin{aligned}
\frac{dx}{dt} &= r_1\left(1-\frac{x}{K_1}-\beta_1 y\right)x\\
\frac{dy}{dt} &= r_2\left(1-\frac{y}{K_2}-\beta_2 x\right)y
\end{aligned}
\tag{1}
$$

Note that if $y(0)=0$, then $y(t)$ remains 0, and the equation for $x(t)$ is

$$
\frac{dx}{dt} = r_1\left(1-\frac{x}{K_1}\right), \tag{2}
$$

which is the familiar logistic equation. Similarly, if $x(0)=0$, then $x(t)$ remains 0 and $y(t)$ is governed by a logistic equation.

Let's do a careful analysis of a specific example, in which $r_1=1$, $K_1=1$, $\beta_1=1$, $r_2=3/4$, $K_2=3/4$, and $\beta_2=2/3$. The differential equations are

$$
\begin{aligned}
\frac{dx}{dt} &= (1-x-y)x\\
\frac{dy}{dt} &= \frac{3}{4}\left(1-\frac{4}{3}y-\frac{2}{3}x\right)y.
\end{aligned}
\tag{3}
$$

We'll find the equilibria, find the linearization at each equlibrium to determine the behavior near each one, and then use the nullclines of (3) to understand what happens in the phase plane "far away" from the equilibria.

Equilibria. To find the equilibria, we must solve

$$
\begin{aligned}
(1-x-y)x &= 0\\
\frac{3}{4}\left(1-\frac{4}{3}y-\frac{2}{3}x\right)y &= 0.
\end{aligned}
\tag{4}
$$

The first equation holds if $x = 0$ or $y = 1 - x$. We consider each case separately in the second equation.

- If $x = 0$, then the second equation of (4) implies $y = 0$ or $y = \frac{3}{4}$.

  So two equilibria are $(0, 0)$ and $(0, 3/4)$.

- If $y = 1 - x$, then the second equation of (4) implies

$$\left(1 - \frac{4}{3}(1-x) - \frac{2}{3}x\right)(1-x) = 0 \implies x = 1 \quad \text{or} \quad x = \frac{1}{2}. \tag{5}$$

  So two equilibria are $(1, 0)$ and $(1/2, 1/2)$.

Linearization at each equilibrium. We have found the following equilibria: $(0, 0)$, $(0, 3/4)$, $(1, 0)$, $(1/2, 1/2)$. We now determine the behavior of (3) near each equilibrium by finding the linearization at each equilibrium. We will need the Jacobian matrix:

$$J = \begin{bmatrix} 1 - 2x - y & -x \\ -\frac{1}{2}y & \frac{3}{4} - 2y - \frac{1}{2}x \end{bmatrix} \tag{6}$$

For each equilibrium, we will find the Jacobian matrix and plot the phase portrait of the linearization. We make two remarks about the phase portraits of the linearized systems:

1. Recall from the notes on "Linearization" that we used the local coordinates (u, v) for the linearization. In a linear system, the scale of the coordinates is not important: if you zoom in on the origin of a linear system, the phase portrait will look exactly the same. So, in the following phase portraits of the linearizations, the ranges on the axis are from -1 to 1. These are *not* the actual x and y ranges.

2. In a population model such as this, $x < 0$ and $y < 0$ are not relevant. However, we will still plot negative values in the linearization. It seems easier to simply plot the linear phase portrait, ignoring the actual meaning until later. Also, recall that the linearized system is expressed in coordinates measured relative to the equilibrium. If a coordinate of the equilibrium is positive, then a negative value of the corresponding local coordinates means that the population is less than the equilibrium value. It does not necessarily mean that the actual population is negative.

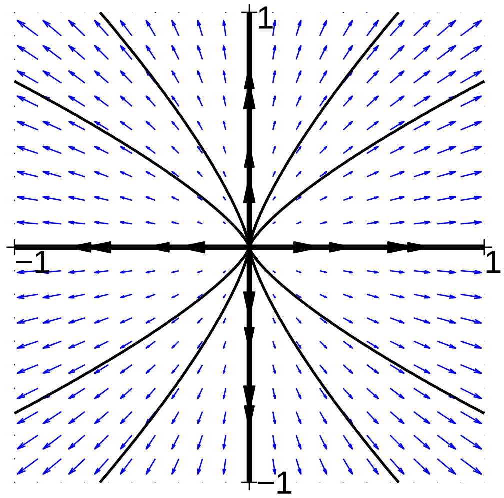
At $(0, 0)$, the Jacobian matrix is

$$J = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{bmatrix}. \tag{7}$$

The eigenvalues are $\lambda_1 = 3/4$ and $\lambda_2 = 1$, with corresponding eigenvectors

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Since $\lambda_1 > 0$ and $\lambda_2 > 0$, the equilibrium $(0, 0)$ is a *source*. The trajectories come out of $(0, 0)$ tangent to the eigenvector $\vec{\mathbf{v}}_1$. The phase portrait of the linearization at $(0, 0)$ is



If we were to “zoom in” on the point $(0,0)$ in (3), this is what the phase portrait would look like.

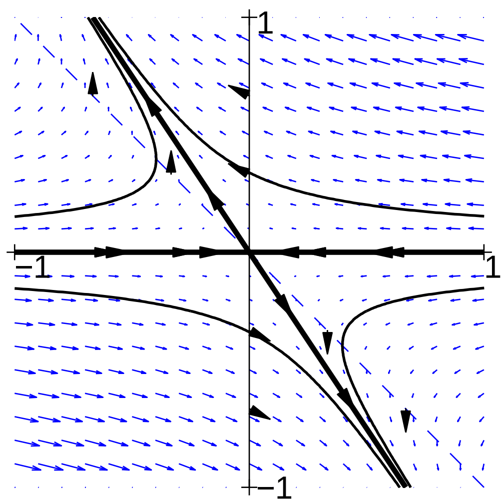
At $(1,0)$, the Jacobian matrix is

$$J = \begin{bmatrix} -1 & -1 \\ 0 & \frac{1}{2} \end{bmatrix}. \tag{8}$$

The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1/2$, with corresponding eigenvectors

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ -3/2 \end{bmatrix}.$$

Since $\lambda_1 < 0$ and $\lambda_2 > 0$, the equilibrium $(1,0)$ is a *saddle point*. The phase portrait of the linearization at $(1,0)$ is



If we were to “zoom in” on the point $(1,0)$ in (3), this is what the phase portrait would look like.

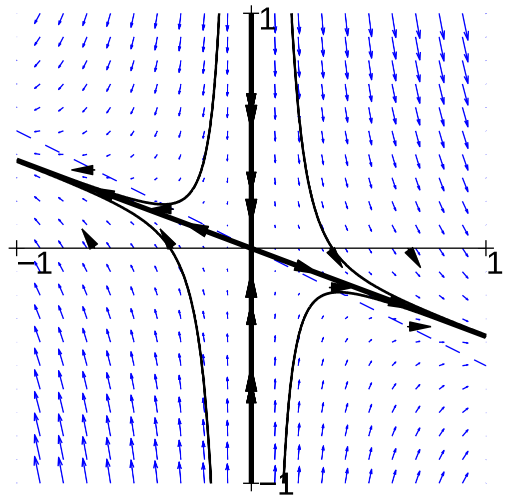
At $(0, 3/4)$, the Jacobian matrix is

$$J = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{3}{8} & -\frac{3}{4} \end{bmatrix}. \tag{9}$$

The eigenvalues are $\lambda_1 = -\frac{3}{4}$ and $\lambda_2 = \frac{1}{4}$, with corresponding eigenvectors

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ -3/8 \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since $\lambda_1 < 0$ and $\lambda_2 > 0$, the equilibrium $(0, 3/4)$ is a *saddle point*. The phase portrait of the linearization at $(0, 3/4)$ is



If we were to "zoom in" on the point $(0, 3/4)$ in (3), this is what the phase portrait would look like.

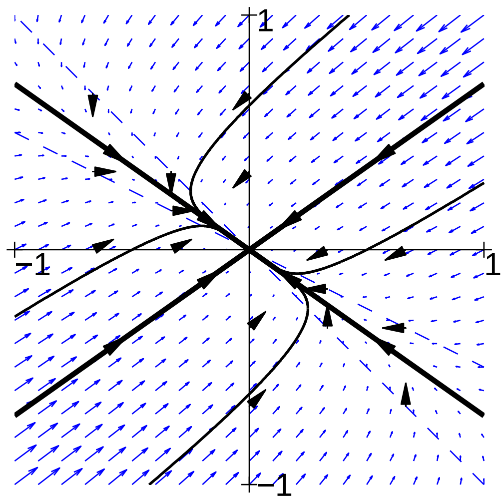
At $(1/2, 1/2)$, the Jacobian matrix is

$$J = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix}. \tag{10}$$

The eigenvalues are $\lambda_1 = \frac{-2-\sqrt{2}}{4} \approx -0.853 < 0$ and $\lambda_2 = \frac{-2+\sqrt{2}}{4} \approx -0.147 < 0$, with corresponding eigenvectors

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{v}}_2 = \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}.$$

Since both eigenvalues are negative, the equilibrium at $(1/2, 1/2)$ is a *sink*. The phase portrait of the linearization at $(1/2, 1/2)$ is



If we were to "zoom in" on the point $(1/2, 1/2)$ in (3), this is what the phase portrait would look like.

Nullclines. Since the eigenvalues at each linearization all had nonzero real parts, each linearization provides a good approximation to the behavior of (3) near the corresponding equilibrium point. However, the local linearizations do not tell us what is happening in the phase plane far from the equilibria. In a planar system such as this, the nullclines can provide useful information about the phase portrait.

The x nullcline is given by

$$(1-x-y)x = 0 \implies x = 0 \quad \text{or} \quad y = 1-x. \tag{11}$$

So $\frac{dx}{dt} = 0$ on the lines $x = 0$ and $y = 1 - x$.[1]

The y nullcline is given by

$$\frac{3}{4}\left(1 - \frac{4}{3}y - \frac{2}{3}x\right)y = 0 \tag{12}$$

which gives the lines

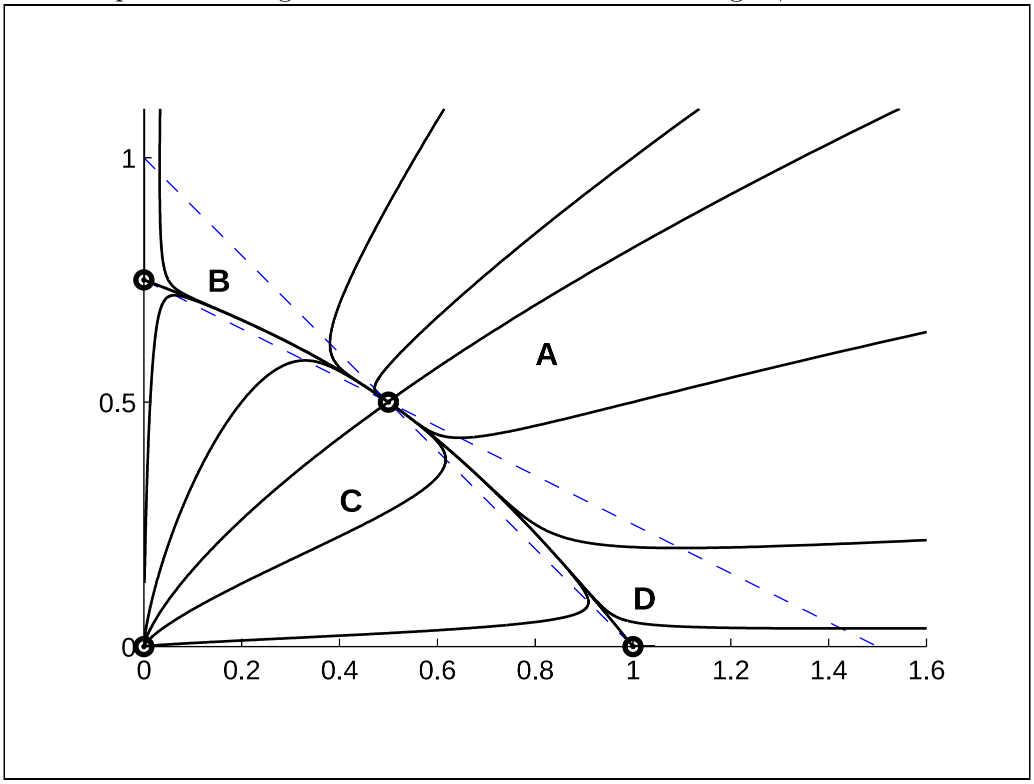
$$y = 0 \quad \text{or} \quad y = \frac{3}{4} - \frac{1}{2}x. \tag{13}$$

On these lines, $\frac{dy}{dt} = 0$.

The nullclines give the points in the plane where $\frac{dx}{dt} = 0$ or $\frac{dy}{dt} = 0$. They form the boundaries of regions in the plane where $\frac{dx}{dt}$ and $\frac{dy}{dt}$ do not change sign. This can be very useful information. For example, if we know that in a certain region, $\frac{dx}{dt} > 0$ and $\frac{dy}{dt} > 0$, we know that all vectors of the vector field point "up and to the right"; this means all trajectories move up and to the right. If this region is in the first quadrant, it implies that all trajectories move away from the origin.

[1] In this example, and in all the planar linear systems that we have studied, the nullclines are straight lines. This is not true in general; nullclines can be curves.

In the following plot, the nullclines (that are not coordinate axes) are plotted with dashed lines, and trajectories are plotted as solid lines. Since our equations give a model for two species, we only include the first quadrant. Negatives values would not be meaningful, so we do not include them.



In the region labeled **A**, $\frac{dx}{dt} < 0$ and $\frac{dy}{dt} < 0$. All trajectories in this region (which extends out to infinity) must move down and to the left. This implies that all these trajectories must either cross the x or y axes, or cross the nullclines that form the boundaries between **A** and **B** and between **A** and **D**. But the coordinate axes are themselves solutions, so solutions that do not start on the coordinate axes can not cross the axes. Thus any trajectory that begins in **A** must eventually cross into **B** or **D** (or, in one special case, converge to the equilibrium at $(1/2, 1/2)$).

Now consider trajectories in **B**. In **B**, we have $\frac{dx}{dt} > 0$ and $\frac{dy}{dt} < 0$. All trajectories in this region move down and to the right. They can not cross either of the nullclines that form the upper and lower boundaries of the region, because the vector field on these nullclines points *into* **B**. Therefore, *all trajectories in* **B** *must converge to* $(1/2, 1/2)$.

A similar argument applies to **D**, where $\frac{dx}{dt} < 0$ and $\frac{dy}{dt} > 0$. In this region, all trajectories move up and to the left, but they can't cross the nullclines, so *all trajectories in* **D** *must converge to* $(1/2, 1/2)$.

Finally, in **C** we have $\frac{dx}{dt} > 0$ and $\frac{dy}{dt} > 0$. All trajectories in this region move up and to the right. Therefore, they must all eventually cross into **B** or **D**, except for the special trajectory in **C** that converges to $(1/2, 1/2)$.

Our conclusion is that any trajectory that begins in the first quadrant, with $x(0) > 0$ and $y(0) > 0$, must converge to $(1/2, 1/2)$. Note that we have concluded this without actually solving the differential equations given in (3). Thus, this population model predicts that the two species will co-exist in a stable equilibrium. (But different parameters values can lead to different conclusions.)