Math 312 Lecture 1
First Order Differential Equations and Applications to Population Models

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In this set of notes we consider some simple models of the growth of a population. The models that we obtain are all first order differential equations.

The simplest assumption for modeling the growth of a population is:

The rate of change of the population is proportional to the current population level.

Let t be time, and let $p(t)$ be the population at time t. For now we'll assume that some convenient system of units is used; we'll have more to say about units later. We also assume that p is "large" in some sense, so that it is reasonable to treat p as a real number, not an integer. If, for example, the population was just three rabbits, the following models would probably not provide good approximations to the actual growth of the population. If the population is measured in thousands of rabbits, then $p(0) = 3.12$ makes sense, and the following models are more reasonable.

We convert the above assumption into an equation involving t and p. The "rate of change of the population" is the derivative, $\frac{dp}{dt}$, and the current population is $p(t)$, so the mathematical version of the above assumption is

$$\frac{dp}{dt} = rp \tag{1}$$

where r is the constant of proportionality. (Note that I have suppressed the argument of p for brevity.) This is a *first order differential equation.* (The *order* of a differential equation is the order of the highest derivative in the equation.)

Frequently, the question we ask is "What is the population at time t if the population is P_0 at time $t = 0$?" That is, we impose the condition that $p(0) = P_0$. This is called an *initial condition*, and the problem of solving the differential equation with a given initial condition is called an *initial value problem.*

To solve this equation, we must find a *function* $p(t)$ that satisfies the equation. In this case, the solution is

$$p(t) = P_0 e^{rt} \tag{2}$$

(Let's check: $\frac{dp}{dt} = rP_0e^{rt} = rp(t)$, so it solves the differential equation, and $p(0) = P_0e^0 = P_0$, so it also satisfies the initial condition.) Thus, if $r > 0$, the simple assumption given above results in *exponential growth.* (If $r < 0$, we obtain *exponential decay.*)

For a population with unlimited resources, exponential growth is a pretty accurate description of what happens. However, no environment can support exponential growth forever. Eventually

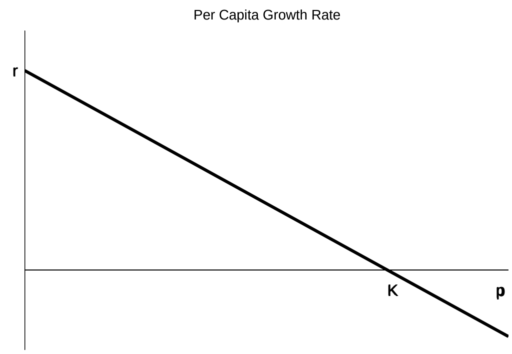

Figure 1: Per capita growth rate for the logistic equation.

the food runs out, or there is simply not enough space for a larger population. Presumably there is a maximum population level that the environment can sustain. This level is called the *carrying capacity* of the environment. If the population is larger than the carrying capacity, overcrowding or a lack of food results in a *decreasing* population. We'll modify the equation to take this into account.

The right side of the equation gives the rate of change of p as a function of p. If we divide the the right side by p, we obtain the *per capita growth rate*, or simply the *growth rate*. For the initial assumption, the growth rate is just the constant r. To incorporate the assumption of a carrying capacity, we will assume that the growth rate depends on p. When p is near zero, we assume that there are plenty of resources for the population to grow, so the growth rate should be near r. As the population becomes bigger, the growth rate should decrease, and it should be zero when the population is at the carrying capacity. If the population exceeds the carrying capacity, the growth rate should be negative.

The simplest formula for such a growth rate is a straight line that has the value r when $p = 0$ and the value 0 when $p = K$, as shown in Figure 1. The equation that we obtain is

$$\frac{dp}{dt} = r\left(1 - \frac{p}{K}\right)p \tag{3}$$

This first order differential equation is commonly called the *logistic equation*. Later we'll see how to find the exact solutions to this equation. For now, we'll learn as much as we can about its solutions without actually solving the equation. To do this, we plot the right side of (3) as a function of p. The graph is shown in Figure 2. We can use the information in Figure 2 to determine the behavior of solutions to (3). Suppose, for example, that the population is initially "small". (In this case, small means "a small fraction of K".) Then the graph in Figure 2 tells us that $\frac{dp}{dt}$ is also small but positive. In other words, the slope of $p(t)$ is small and positive. This means that $p(t)$ is an increasing function of t, so in a little while, p will be larger. Looking back to Figure 2, we see that this means $\frac{dp}{dt}$ will be larger than it was before, and therefore the slope of $p(t)$ has increased. So initially, $p(t)$ will grow faster and faster. This is not surprising, since when p is small, p^2 is very small, and if we ignore the p^2 term in the logistic equation, we obtain $\frac{dp}{dt} \approx rp$. So when the population is small, the growth is almost exponential.

Eventually the population will reach $K/2$. This is the population level at which the population grows the fastest. When the population reaches this level, further increases in the population result in *lower* growth rates. For example, if $p(t) = 0.75K$ at some time t, we see in Figure 2 that $\frac{dp}{dt}$

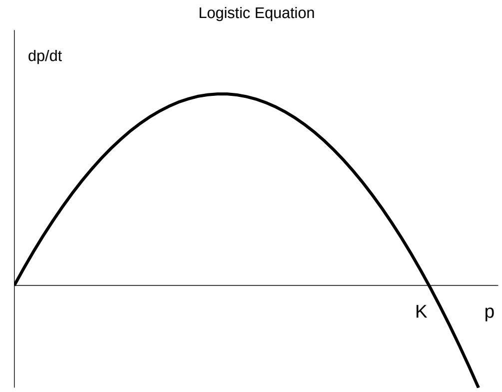

Figure 2: A plot of $\frac{dp}{dt}$ as a function of p for the logistic equation (3).

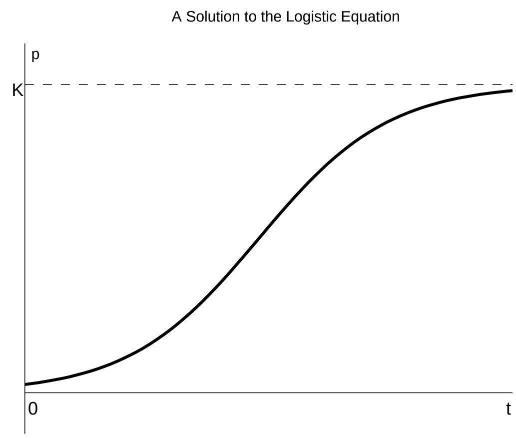

Figure 3: The graph of $p(t)$, a solution to the logistic equation (3).

is positive, but it is decreasing as p increases. In the next moment, p will be larger, but then the slope of $p(t)$ will be smaller. As p gets closer and closer to K, the slope gets smaller and smaller. In fact, $p(t)$ will approach K asymptotically, but never reach it. A graph of a solution to the logistic equation (3) is shown in Figure 3.

Question 1 *What happens if the initial population is greater than K?*

Note that if the inital population is exactly K, then $\frac{dp}{dt} = 0$. The constant function $p(t) = K$ is an exact solution to the differential equation. We call a constant solution an *equilibrium solution.*

Question 2 *Does the logistic equation* (3) *have any other equilibrium solutions?*

The method for analyzing solutions to the logistic equation can be applied to any first order equation of the form

$$\frac{dy}{dt} = f(y) \tag{4}$$

where $f(y)$ is a given function. In particular, we assume that t does *not* appear explicitly in f. Such an equation is called *autonomous.* When t appears explicity in the right side of the equation, we say the equation is *nonautonomous.* For example, the equation

$$\frac{dp}{dt} = (r + a\sin(\omega t))p \tag{5}$$

is nonautonomous, since t appears explicitly in the right side of the equation (in the argument of the sine function). In this case, the explicit time dependence might model a growth rate with seasonal dependence.

The basic procedure to analyze an autonomous first order differential equation is to sketch the graph of $f(y)$, identify the equilibrium solutions (where $f(y) = 0$), and determine the behavior of non-equilibrium solutions based on the graph of $f(y)$.

Question 3 *Consider the differential equation*

$$\frac{dy}{dt} = y(y-1)(y-2)$$

(a) Find the equilibrium solutions.
(b) Sketch $\frac{dy}{dt}$ as a function of y.
(c) In one set of axes, sketch several solutions $y(t)$. Include the equilibrium solutions, and several more non-equilibrium solutions to show all the possible behaviors. (Since this is not necessarily a population model, you should consider negative values of y as well as positive.)