Math 312 Lecture Notes
Linear Two-dimensional Maps

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In these notes, we consider the linear maps of the plane

$$\begin{aligned} x_{n+1} &= a_{11}x_n + a_{12}y_n \\ y_{n+1} &= a_{21}x_n + a_{22}y_n \end{aligned} \tag{1}$$

with a given starting point (x_0, y_0), or equivalently,

$$\vec{\mathbf{x}}_{n+1} = A\vec{\mathbf{x}}_n, \quad \text{where} \quad \vec{\mathbf{x}}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{2}$$

and the starting vector is $\vec{\mathbf{x}}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

Solving the System

We note if $\vec{\mathbf{x}}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \equiv \vec{\mathbf{0}}$, then $\vec{\mathbf{x}}_n = \vec{\mathbf{0}}$ for all $n > 0$ is a solution to (2). Such a constant solution is called a *fixed point* of the map.

More generally, we can "solve" this system by simply iterating the map:

$$\begin{aligned} \vec{\mathbf{x}}_1 &= A\vec{\mathbf{x}}_0 \\ \vec{\mathbf{x}}_2 &= A\vec{\mathbf{x}}_1 = A^2\vec{\mathbf{x}}_0 \\ \vec{\mathbf{x}}_3 &= A\vec{\mathbf{x}}_2 = A^3\vec{\mathbf{x}}_0 \\ &\vdots \\ \vec{\mathbf{x}}_n &= A^n\vec{\mathbf{x}}_0 \end{aligned} \tag{3}$$

This is a solution, but it doesn't tell us much about the behavior of the solutions. A lot of information is hidden in A^n.

An alternative approach is to use a procedure similar to the method we used to solve linear systems of differential equations. In the case of a linear map, we propose a solution of the form

$$\vec{\mathbf{x}}_n = \lambda^n \vec{\mathbf{v}} \tag{4}$$

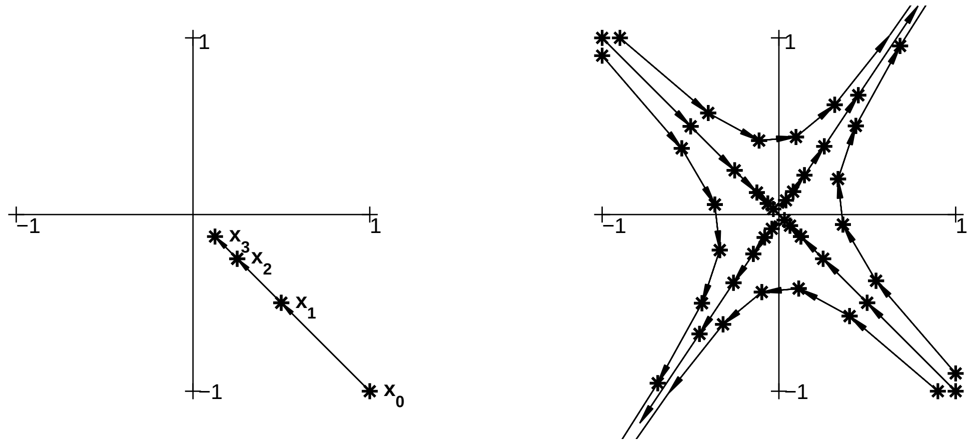

Figure 1: The plot on the left shows several iterations of the solution $\vec{\mathbf{x}}_n = \lambda_1^n \vec{\mathbf{v}}_1$ of the linear map for A given in (7). The trajectory begins at $\vec{\mathbf{x}}_0 = \vec{\mathbf{v}}_1$. The plot on the right shows several more trajectories of the same system.

where λ is a number and $\vec{\mathbf{v}}$ is a constant vector. We already know that $\vec{\mathbf{x}}_n \equiv \vec{\mathbf{0}}$ is a solution, so without loss of generality we assume that $\lambda \neq 0$ and $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$.

By substituting this "guess" into (2), we obtain

$$\lambda^{n+1}\vec{\mathbf{v}} = A\lambda^n \vec{\mathbf{v}} \tag{5}$$

or, after canceling a factor of λ and rearranging,

$$A\vec{\mathbf{v}} = \lambda \vec{\mathbf{v}}. \tag{6}$$

This is the familiar *eigenvalue problem* for the matrix A. To solve (2), we must find the eigenvalues (λ) and the corresponding eigenvectors ($\vec{\mathbf{v}}$) of A. If λ is an eigenvalue of A and $\vec{\mathbf{v}}$ is a corresponding eigenvector, then (4) is a solution to (2).

Let's consider what such a solution looks like in the (x, y) plane. Consider, for example,

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{3}{4} & \frac{5}{4} \end{bmatrix} \tag{7}$$

The eigenvalues of A are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{7}{4}$, with eigenvectors $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{\mathbf{v}}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, respectively. So two solutions are $(1/2)^n \vec{\mathbf{v}}_1$ and $(7/4)^n \vec{\mathbf{v}}_2$. The plot on the left in Figure 1 shows the first three iterations of the solutions $(1/2)^n \vec{\mathbf{v}}_1$. Since $(1/2)^n \to 0$ and n increases, further iterations will converge to the origin. The iterates in the second solution, $(7/4)^n \vec{\mathbf{v}}_2$, remain on the line $y = (3/2)x$, but in this case they move further and further *away* from the origin, since $7/4 > 1$.

Because the map is linear, we can form the *general solution* by taking linear combinations of these two special solutions. That is, at least when λ_1 and λ_2 are real and distinct eigenvalues, the general solution is

$$\vec{\mathbf{x}}_n = c_1 \lambda_1^n \vec{\mathbf{v}}_1 + c_2 \lambda_2^n \vec{\mathbf{v}}_2. \tag{8}$$

The constant c_1 and c_2 are chosen so that the initial condition is satisfied. That is,

$$c_1 \vec{\mathbf{v}}_1 + c_2 \vec{\mathbf{v}}_2 = \vec{\mathbf{x}}_0. \tag{9}$$

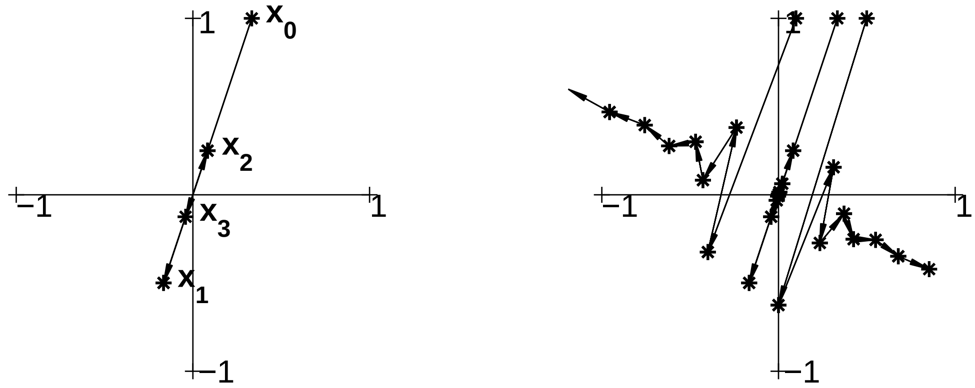

Figure 2: The plot on the left shows several iterations of the solution $\vec{\mathbf{x}}_n = \lambda_1^n \vec{\mathbf{v}}_1$ of the linear map for A given in (10). The trajectory begins at $\vec{\mathbf{x}}_0 = \vec{\mathbf{v}}_1$. The plot on the right shows two more trajectories of the same system.

Figure 1 shows several trajectories of the map with A given in (7). The trajectories shown in Figure 1 might suggest that the trajectories of linear maps behave the same as trajectories of linear systems of differential equations, only with discrete jumps instead of smooth curves. However, maps actually have more possible behaviors. Consider, for example, the map with

$$A = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{3}{4} & -\frac{1}{4} \end{bmatrix} \tag{10}$$

This matrix has eigenvalues $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = \frac{5}{4}$, with eigenvectors $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$ and $\vec{\mathbf{v}}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, respectively. The first few iterations of the solution $\lambda_1^n \vec{\mathbf{v}}_1$ are shown on the left in Figure 2. Because $\lambda_1 < 0$, λ_1^n alternates sign, and the trajectory jumps from one side of the origin to the other as n increases. Two more trajectories for this system are shown on the right in Figure 2. Note that after a few iterations, the contribution of $c_1 \lambda_1^n \vec{\mathbf{v}}_1$ becomes very small (since $\lambda_1^n \to 0$), and the solutions are eventually dominated by $c_2 \lambda_2^n \vec{\mathbf{v}}_2$. This means the trajectories converge to the line $y = -x/2$ as n increases, and since $\lambda_2 > 1$, the iterations diverge from $\vec{\mathbf{0}}$ as n increases.

Stability. Suppose A has real and distinct eigenvalues λ_1 and λ_2, with eigenvectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$.

- If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then $\vec{\mathbf{x}}_n \to \vec{\mathbf{0}}$ as n increases. We say $\vec{\mathbf{0}}$ is a *sink* or *attractor*; also $\vec{\mathbf{0}}$ is *asymptotically stable*.
- If one eigenvalue, say λ_1 has magnitude less than one and the other has magnitude greater than one, then we call $\vec{\mathbf{0}}$ a saddle. (Both the examples shown in Figures 1 and 2 are saddles.) A saddle is *unstable*, because there are trajectories beginning arbitrarily close of $\vec{\mathbf{0}}$ that diverge from $\vec{\mathbf{0}}$.
- If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, then $\vec{\mathbf{0}}$ is a *source* or *repellor*. (Of course, a source is also unstable.)

Complex Eigenvalues. Here we consider the solutions when the eigenvalues of A are complex. Recall that complex eigenvalues of a A matrix must occur as complex conjugate pairs. The following discussion is quite similar to how we developed real-valued solutions for linear systems of differential equations with complex eigenvalues.

Let $\lambda_1 = \mu + i\omega$, with eigenvector $\vec{\mathbf{v}}_1 = \vec{\mathbf{a}} + i\vec{\mathbf{b}}$. A *complex-valued* solution is $\vec{\mathbf{x}}_n = (\lambda_1)^n \vec{\mathbf{v}}_1$. We'll write this as the sum of a real and imaginary part; the real and imaginary parts of this solution are also solutions, so they will give us a *real-valued* set of solutions with which we can create the general solution.

Recall *Euler's Identity*: $e^{i\theta} = \cos\theta + i\sin\theta$. We use this to write

$$\begin{aligned} \mu + i\omega &= \sqrt{\mu^2+\omega^2}\left(\frac{\mu}{\sqrt{\mu^2+\omega^2}} + i\frac{\omega}{\sqrt{\mu^2+\omega^2}}\right) \\ &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta}, \end{aligned} \tag{11}$$

where $r = |\lambda_1| = \sqrt{\mu^2+\omega^2}$ and $\theta = \tan^{-1}\left(\frac{\omega}{\mu}\right)$. (The *magnitude* of a complex number $\lambda_1 = \mu + i\omega$ is $|\lambda_1| = \sqrt{\mu^2+\omega^2}$. If we think of the complex number as the point (μ, ω) in Cartesian coordinates, then r and θ are the polar coordinates of the point.) Then

$$\begin{aligned} \lambda_1^n &= (re^{i\theta})^n \\ &= r^n e^{in\theta} \\ &= r^n(\cos(n\theta) + i\sin(n\theta)) \end{aligned} \tag{12}$$

and

$$\begin{aligned} \lambda_1^n \vec{\mathbf{v}}_1 &= r^n(\cos(n\theta) + i\sin(n\theta))(\vec{\mathbf{a}} + i\vec{\mathbf{b}}) \\ &= r^n(\vec{\mathbf{a}}\cos(n\theta) - \vec{\mathbf{b}}\sin(n\theta)) + ir^n(\vec{\mathbf{a}}\sin(n\theta) + \vec{\mathbf{b}}\cos(n\theta)) \end{aligned} \tag{13}$$

The real and imaginary parts of this solution are

$$u_n = r^n(\vec{\mathbf{a}}\cos(n\theta) - \vec{\mathbf{b}}\sin(n\theta)) \quad \text{and} \quad w_n = r^n(\vec{\mathbf{a}}\sin(n\theta) + \vec{\mathbf{b}}\cos(n\theta)), \tag{14}$$

respectively. Each of these is solution to linear map, and we can use these to write the general solution as

$$\vec{\mathbf{x}}_n = c_1 r^n(\vec{\mathbf{a}}\cos(n\theta) - \vec{\mathbf{b}}\sin(n\theta)) + c_2 r^n(\vec{\mathbf{a}}\sin(n\theta) + \vec{\mathbf{b}}\cos(n\theta)) \tag{15}$$

Note that the terms in parentheses give vectors that rotate by θ for each increase of n by one. We have the following possibilities for the behavior of $\vec{\mathbf{x}}_n$:

- If $r < 1$, then $r^n \to 0$ as n increases, and therefore $\vec{\mathbf{x}}_n \to \vec{\mathbf{0}}$. In this case, we say that $\vec{\mathbf{0}}$ is a *spiral sink*. An example is shown on the left in Figure 3.
- If $r = 1$, then in the long run $\vec{\mathbf{x}}_n$ does not approach $\vec{\mathbf{0}}$ or go off to infinity. Instead, it remains on an ellipse. (However, $\vec{\mathbf{x}}_n$ is *not* periodic unless $\frac{\theta}{2\pi}$ is a rational number.) When $r = 1$ we say that $\vec{\mathbf{0}}$ is a *center*. Examples are shown in Figure 4.
- If $r > 1$, then r^n grows exponentially, so $\vec{\mathbf{x}}_n$ spirals away from the origin. We say that $\vec{\mathbf{0}}$ is a *spiral source*. An example is shown on the right in Figure 3.

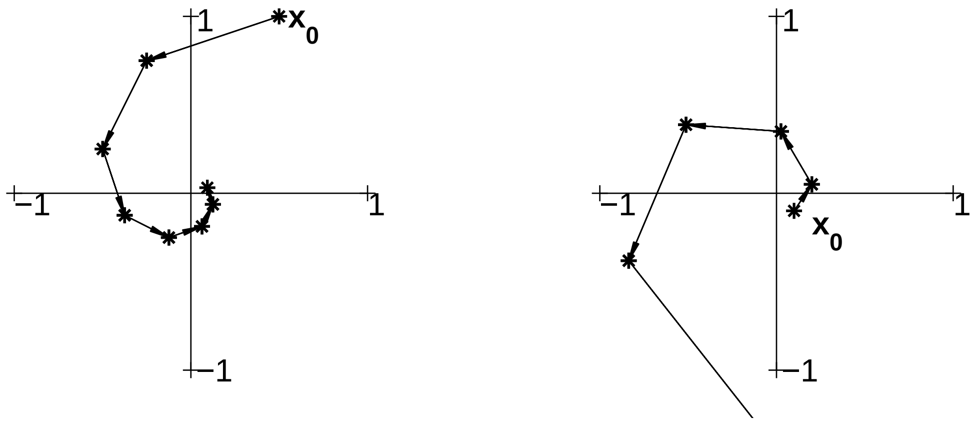

Figure 3: Examples with complex eigenvalues. On the left: $A = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$, $\lambda = 1/2 \pm i/2$, $r = \sqrt{1/2} < 1$. On the right: $A = \begin{bmatrix} 1/2 & -3/2 \\ 3/2 & 1 \end{bmatrix}$, $\lambda = 3/4 \pm i\sqrt{35}/4$, $r = \sqrt{11/4} > 1$.

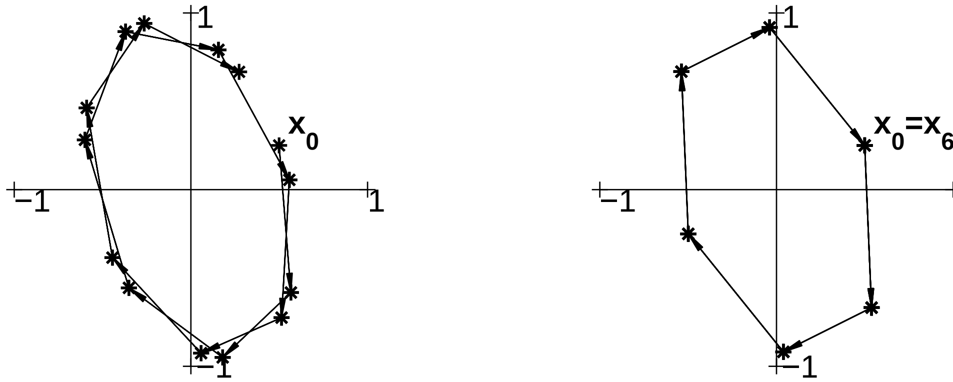

Figure 4: Examples with complex eigenvalues. On the left: $A = \begin{bmatrix} 13/15 & 8/15 \\ -4/3 & 1/3 \end{bmatrix}$, $\lambda = 3/5 \pm i4/5$, $r = 1$. On the right: $A = \frac{1}{6}\begin{bmatrix} 3+\sqrt{3} & 2\sqrt{3} \\ -5\sqrt{3} & 3-\sqrt{3} \end{bmatrix}$, $\lambda = 1/2 \pm i\sqrt{3}/2$, $r = 1$, $\theta = \pi/3$.

Summary for the 2D Linear Maps.

We summarize here the general solution and stability results for the two-dimensional linear maps (2). Let λ_1 and λ_2 be the eigenvalues of A, with eigenvectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$. If the eigenvalues are complex, we assume $\lambda_1 = \mu + i\omega$, where μ and ω are real numbers, and $\omega > 0$. We also let $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ be the real vectors such that $\vec{\mathbf{v}}_1 = \vec{\mathbf{a}} + i\vec{\mathbf{b}}$. We only consider the case in which the eigenvalues are distinct.

General Solution:

- If the eigenvalues of A are real and distinct, then the general solution is

$$\vec{\mathbf{x}}_n = c_1\lambda_1^n\vec{\mathbf{v}}_1 + c_2\lambda_2^n\vec{\mathbf{v}}_2 \tag{16}$$

- If the eigenvalues are complex, then

$$\vec{\mathbf{x}}_n = c_1 r^n(\vec{\mathbf{a}}\cos(n\theta) - \vec{\mathbf{b}}\sin(n\theta)) + c_2 r^n(\vec{\mathbf{a}}\sin(n\theta) + \vec{\mathbf{b}}\cos(n\theta)) \tag{17}$$

Stability:

- If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then $\vec{\mathbf{0}}$ is a *sink* or *attractor*; it is *asymptotically stable.*
- If either $|\lambda_1| > 1$ or $|\lambda_2| > 1$, then $\vec{\mathbf{0}}$ is *unstable.*
- If both $|\lambda_1| > 1$ and $|\lambda_2| > 1$, then $\vec{\mathbf{0}}$ is a *source* or *repellor.*

Affine 2D Systems

We point out here that it is no harder to solve the system

$$\vec{\mathbf{x}}_{n+1} = A\vec{\mathbf{x}}_n + \vec{\mathbf{g}}, \tag{18}$$

where $\vec{\mathbf{g}}$ is a constant vector. This is an example of a *nonhomogeneous* linear system. It is also called an *affine system*.

First, we find the fixed point $\vec{\mathbf{x}}^*$ of the map. (In the following, we will assume that the matrix $A - I$ is invertible. The case where $A - I$ is singular is left as an exercise.) The fixed point satisfies the equation

$$\begin{aligned} A\vec{\mathbf{x}}^* + \vec{\mathbf{g}} &= \vec{\mathbf{x}}^* \\ (A - I)\vec{\mathbf{x}}^* &= -\vec{\mathbf{g}} \\ \vec{\mathbf{x}}^* &= -(A - I)^{-1}\vec{\mathbf{g}}. \end{aligned} \tag{19}$$

Then define $\vec{\mathbf{y}}_n$ so $\vec{\mathbf{x}}_n = \vec{\mathbf{x}}^* + \vec{\mathbf{y}}_n$, and (18) becomes

$$\begin{aligned} \vec{\mathbf{x}}^* + \vec{\mathbf{y}}_{n+1} &= A\vec{\mathbf{x}}^* + A\vec{\mathbf{y}}_n + \vec{\mathbf{g}} \\ \vec{\mathbf{y}}_{n+1} &= A\vec{\mathbf{y}}_n + (A - I)\vec{\mathbf{x}}^* + \vec{\mathbf{g}} \\ &= A\vec{\mathbf{y}}_n + (A - I)\left(-(A - I)^{-1}\vec{\mathbf{g}}\right) + \vec{\mathbf{g}} \\ &= A\vec{\mathbf{y}}_n - \vec{\mathbf{g}} + \vec{\mathbf{g}} \\ &= A\vec{\mathbf{y}}_n. \end{aligned} \tag{20}$$

So by changing coordinates to $\vec{\mathbf{y}}_n$, we obtain the linear system

$$\vec{\mathbf{y}}_{n+1} = A\vec{\mathbf{y}}_n. \tag{21}$$

We saw the same idea when we added a constant vector to a linear system of differential equations. Adding the constant vector to the right side of the equation moves the equilibrium or fixed point, but the dynamics around that point are the same as the linear system without the constant vector added.