Math 312 Lecture Notes
Linearization

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These notes discuss *linearization*, in which a linear system is used to approximate the behavior of a nonlinear system. We will focus on two-dimensional systems, but the techniques used here also work in n dimensions.

We have seen two broad classes of equations that can be used to model systems that change over time. If we assume time is continuous, we obtain differential equations, and if we use discrete time, we obtain maps.

- **Differential equations.** A system of two (autonomous) differential equations has the form

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{aligned} \tag{1}$$

  The constant solutions to this system are called the equilibria. They satisfy the equation

$$f(x^*,y^*) = 0, \quad g(x^*,y^*) = 0. \tag{2}$$

- **Discrete maps.** A two-dimensional map has the form

$$\begin{aligned} x_{n+1} &= f(x_n,y_n) \\ y_{n+1} &= g(x_n,y_n) \end{aligned} \tag{3}$$

  The constant solutions to this system are called the fixed points of the map. They satisfy the equation

$$f(x^*,y^*) = x^*, \quad g(x^*,y^*) = x^*. \tag{4}$$

In either case, if the system is *linear with constant coefficients*, we have learned how to solve it. Unfortunately, most problems that arise in the real world are not linear, and in most cases, nonlinear systems can not be "solved"–there is typically no method for deriving a solution to the equations.

When confronted with a nonlinear problem, we usually must be satisfied with an approximate solution. One method to find approximate solutions is *linearization.* This method is quite general; in these notes, we will look at the linearization of the equations near a constant solution.

Recall from calculus that the linearization (or tangent plane approximation) of $f(x, y)$ at a point (x^*, y^*) is

$$f(x, y) \approx f(x^*, y^*) + f_x(x^*, y^*)(x - x^*) + f_y(x^*, y^*)(y - y^*), \tag{5}$$

where $f_x(x, y)$ is the partial derivative of f with respect to x. This is also written $\frac{\partial f}{\partial x}$.

Linearization at an equilibrium point of a system of differential equations. By replacing $f(x, y)$ in (1) with its linear approximation near (x^*, y^*), we obtain

$$\frac{dx}{dt} = f(x^*, y^*) + f_x(x^*, y^*)(x - x^*) + f_y(x^*, y^*)(y - y^*). \tag{6}$$

If (x^*, y^*) is an equilibrium of (1), we have $f(x^*, y^*) = 0$, so we can drop that term on the right. The linear approximation of $g(x, y)$ near (x^*, y^*) gives a corresponding equation for $\frac{dy}{dt}$.

Define new coordinates $u = x - x^*$, $v = y - y^*$. The (u, v) coordinates are coordinates measured relative to (x^*, y^*). In the (u, v) coordinates, the equilibrium is at the origin. Since x^* and y^* are constants, we have $\frac{du}{dt} = \frac{dx}{dt}$, and $\frac{dv}{dt} = \frac{dy}{dt}$. Writing the linear approximations in terms of u and v gives us

$$\begin{aligned} \frac{du}{dt} &= f_x(x^*, y^*)u + f_y(x^*, y^*)v \\ \frac{dv}{dt} &= g_x(x^*, y^*)u + g_y(x^*, y^*)v \end{aligned} \tag{7}$$

This is the *linearization of* (1) *at* (x^*, y^*). By defining $\vec{\mathbf{u}} = \begin{bmatrix} u \\ v \end{bmatrix}$, we can write this in matrix form as

$$\frac{d\vec{\mathbf{u}}}{dt} = J\vec{\mathbf{u}}, \tag{8}$$

where

$$J = \begin{bmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{bmatrix} \tag{9}$$

is called the *Jacobian matrix*.

Linearization of a map at a fixed point. We can also find the linearizaton of a discrete map at a fixed point (x^*, y^*). In this case, replacing $f(x, y)$ with its tangent plane approximation at (x^*, y^*) converts the first equation in (3) to

$$x_{n+1} = f(x^*, y^*) + f_x(x^*, y^*)(x_n - x^*) + f_y(x^*, y^*)(y_n - y^*). \tag{10}$$

Similar to the previous case, we define $u_n = x_n - x^*$ and $v_n = y_n - y^*$. Then $x_{n+1} = u_{n+1} + x^*$, and (10) becomes

$$u_{n+1} + x^* = f(x^*, y^*) + f_x(x^*, y^*)u_n + f_y(x^*, y^*)v_n \tag{11}$$

At a fixed point, $f(x^*, y^*) = x^*$, so we can cancel the x^* on the left with the $f(x^*, y^*)$ on the right. Applying the same arguments to the equation for y_{n+1} results in the system

$$\begin{aligned} u_{n+1} &= f_x(x^*, y^*)u_n + f_y(x^*, y^*)v_n \\ v_{n+1} &= g_x(x^*, y^*)u_n + g_y(x^*, y^*)v_n \end{aligned} \tag{12}$$

This is the linearization of (3) at (x^*, y^*). By defining the vector $\vec{\mathbf{u}}_n = \begin{bmatrix} u_n \\ v_n \end{bmatrix}$, we can write the system in matrix form as

$$\vec{\mathbf{u}}_{n+1} = J\vec{\mathbf{u}}_n, \tag{13}$$

where J is the Jacobian matrix given in (9).

What does the linearization tell us about the original system? Equations (8) and (13) are the linear approximations to (1) and (3), respectively. But how "good" are the approximations? We have the following result:

- *For differential equations:* If the real parts of both eigenvalues are nonzero, then the behavior of the system (1) near (x^*, y^*) is qualitatively the same as the behavior of the linear approximation (8). The classification of the equilibrium in the nonlinear system is the same as the classification of the origin in the linearization.

- *For maps:* If neither eigenvalue has magnitude equal to 1, then the behavior of the system (3) near (x^*, y^*) is qualitatively the same as the behavior of the linear approximation (13). The classification of the fixed point of the nonlinear map is the same as the classification of the origin in the linearization.

These are the cases where the linear approximation contains enough information to determine the actual behavior of the nonlinear system.

In a one-dimensional map $x_{n+1} = f(x_n)$, with a fixed point x^*, the Jacobian "matrix" is simply $f'(x^*)$. We saw examples in the lecture notes on one-dimensional maps that showed why we could not determine the stability of a fixed point based on just the linearization in the case $|f'(x^*)| = 1$. The above results are a generalization of that phenomena to higher dimensions.

Example. The system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 3x - y^2 \\ \frac{dy}{dt} &= \sin(y) - x \end{aligned} \tag{14}$$

has two equilibria, one of which is $(0, 0)$. A phase portrait (generated with PPLANE) is shown in Figure 1. The Jacobian matrix is

$$J = \begin{bmatrix} 3 & -2y \\ -1 & \cos(y) \end{bmatrix} \tag{15}$$

and at $(0, 0)$, this is

$$J = \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix}. \tag{16}$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$. Both $\lambda_1 > 0$ and $\lambda_2 > 0$, so the origin in the linearization is a *source*. Since the real part of both eigevalues is nonzero, we conclude that the equilibrium $(0, 0)$ of the original nonlinear equations is also a source. *Near* $(0, 0)$, the linearization provides a good approximation to the nonlinear system. The image on the left in Figure 2 shows the phase portrait of (14) near $(0, 0)$, and the image on the right is the phase portrait of the linearization at $(0, 0)$.

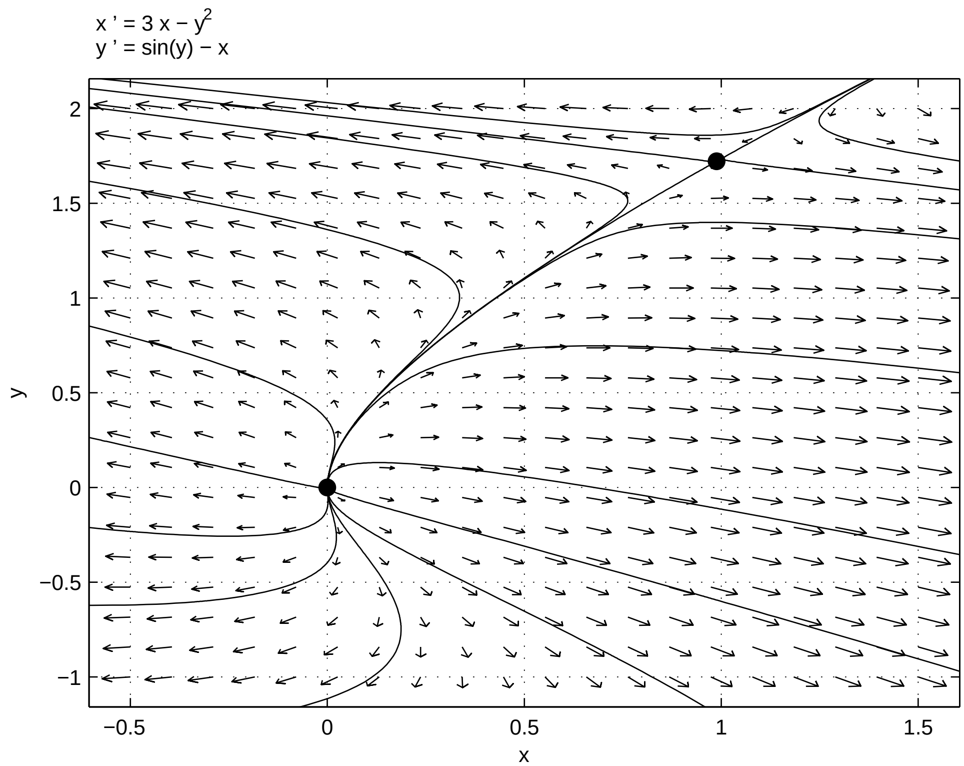

Figure 1: Phase portrait for system (14).

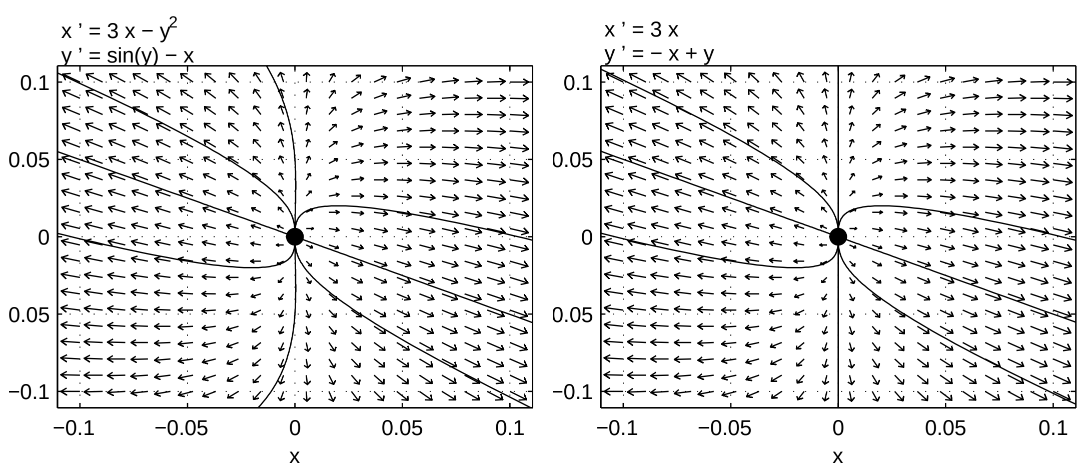

Figure 2: On the left is the phase portrait of (14) near $(0, 0)$, and on the right is the phase portrait of the linearization at $(0, 0)$. They are almost the same. If we zoomed in closer, they would appear even more similar.

Example. The system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 2x - y - x^2 \\ \frac{dy}{dt} &= x - 2y + y^2 \end{aligned} \tag{17}$$

has equilibria at $(0, 0)$ and $(1, 1)$. The Jacobian matrix is

$$J = \begin{bmatrix} 2 - 2x & -1 \\ 1 & -2 + 2y \end{bmatrix} \tag{18}$$

At $(0, 0)$, this is

$$J = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} \tag{19}$$

This matrix has eigenvalues $\lambda_1 = -\sqrt{3}$ and $\lambda_2 = \sqrt{3}$, so the origin of the linearized system is a saddle point. Both eigenvalues are real and nonzero, so we conclude that the equilibrium $(0, 0)$ of the nonlinear system is also a saddle point.

At $(1, 1)$, the Jacobian matrix is

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{20}$$

This matrix has eigenvalues $\lambda = \pm i$, so the linearization results in a *center*. Because the real parts of the eigenvalues are zero, we can *not* conclude that $(1, 1)$ is actually a center in the nonlinear system. Trajectories near $(1, 1)$ will rotate around $(1, 1)$, but the linearization can not tell us if these trajectories actually form closed curves. The trajectories might, in fact, slowly spiral towards or away from $(1, 1)$.

Example. Consider the map

$$\begin{aligned} x_{n+1} &= (1 - x_n)x_n + by_n \\ y_{n+1} &= \frac{y_n}{2} + x_n \end{aligned} \tag{21}$$

where b is a constant. This is a map of the form shown in (3), with

$$f(x, y) = (1 - x)x + by \quad \text{and} \quad g(x, y) = \frac{y}{2} + x. \tag{22}$$

To find the fixed points, we must solve

$$(1 - x)x + by = x, \quad \text{and} \quad \frac{y}{2} + x = y. \tag{23}$$

From the second equation we have $y = 2x$; putting this into the first equation leads to

$$x^2 - 2bx = 0 \tag{24}$$

and the solutions are

$$x = 0 \quad \text{or} \quad x = 2b. \tag{25}$$

Thus the fixed points are $(0,0)$ and $(2b, 4b)$. In this example, we'll focus on the behavior near $(0,0)$.

The Jacobian matrix at a fixed point (x^*, y^*) is

$$J = \begin{bmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{bmatrix} = \begin{bmatrix} 1 - 2x^* & b \\ 1 & \frac{1}{2} \end{bmatrix} \tag{26}$$

At the fixed point $(0,0)$, we find

$$J = \begin{bmatrix} 1 & b \\ 1 & \frac{1}{2} \end{bmatrix} \tag{27}$$

The eigenvalues of the Jacobian are

$$\lambda_1 = \frac{3}{4} - \frac{1}{2}\sqrt{\frac{1}{4} + 4b}, \quad \lambda_2 = \frac{3}{4} + \frac{1}{2}\sqrt{\frac{1}{4} + 4b}. \tag{28}$$

The first thing to determine is whether the eigenvalues are complex or real. The eigenvalues are complex if

$$\frac{1}{4} + 4b < 0 \implies b < -\frac{1}{16}. \tag{29}$$

So we have complex eigenvalues if $b < -\frac{1}{16}$ and real eigenvalues if $b \geq -\frac{1}{16}$. We treat each case separately.

When $b < -\frac{1}{16}$, the eigenvalues are

$$\lambda = \frac{3}{4} \pm \frac{i}{2}\sqrt{-\frac{1}{4} - 4b} \tag{30}$$

To classify the fixed point and determine its stability, we must determine whether the magnitude of the eigenvalues are greater than or less than one. To do this, we'll find the value(s) of b, if any, where $|\lambda| = 1$. (Note that this is equivalent to $|\lambda|^2 = 1$.) We have

$$\begin{aligned} 1 &= |\lambda|^2 \\ &= \left(\frac{3}{4}\right)^2 + \left(\frac{1}{2}\sqrt{-\frac{1}{4} - 4b}\right)^2 \\ &= \frac{1}{2} - b \end{aligned} \tag{31}$$

which gives $b = -\frac{1}{2}$. So we have the following:

- If $b < -\frac{1}{2}$, then $|\lambda| > 1$, and $(0,0)$ is a spiral source.
- If $-\frac{1}{2} < b < -\frac{1}{16}$, then $|\lambda| < 1$, and $(0,0)$ is a spiral sink.

Next we consider $b > -\frac{1}{16}$, where the eigenvalues are real. From (28) we observe that $\lambda_1 < \frac{3}{4}$, and $\lambda_2 > \frac{3}{4}$, so we only need to determine if $\lambda_1 < -1$ or $\lambda_2 > 1$. First, consider

$$\begin{aligned} \lambda_1 &= -1 \\ \frac{3}{4} - \frac{1}{2}\sqrt{\frac{1}{4} + 4b} &= -1 \\ \frac{1}{4} + 4b &= \frac{49}{4} \\ b &= 3 \end{aligned} \tag{32}$$

So $\lambda_1 < -1$ if $b > 3$.

Next consider

$$\begin{aligned}\lambda_2 &= 1 \\ \frac{3}{4} + \frac{1}{2}\sqrt{\frac{1}{4} + 4b} &= 1 \\ \frac{1}{4} + 4b &= \frac{1}{4} \\ b &= 0\end{aligned} \tag{33}$$

So $\lambda_2 < 1$ if $-\frac{1}{16} < b < 0$. Thus, we have

- If $-\frac{1}{16} < b < 0$, we have $-1 < \lambda_1 < 1$ and $\frac{3}{4} < \lambda_2 < 1$, so $(0,0)$ is a sink.
- If $0 < b < 3$, then $-1 < \lambda_1 < 1$ but $\lambda_2 > 1$, so $(0,0)$ is a saddle.
- If $b > 3$, then $\lambda_1 < -1$ and $\lambda_2 > 1$, so $(0,0)$ is a source.

The five bulleted statements above give all the cases for the fixed point $(0,0)$. Figure 3 shows trajectories near the fixed point $(0,0)$ for several values of β.

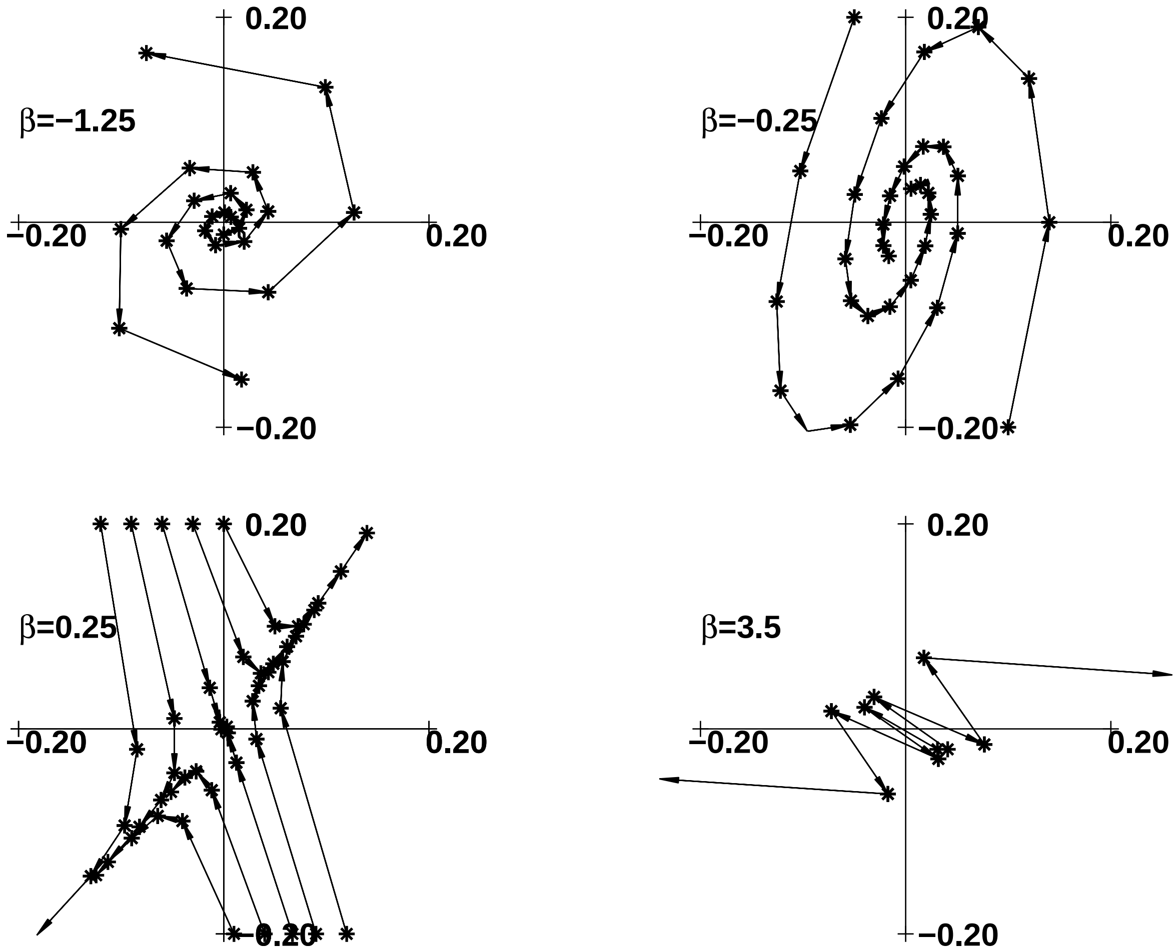

Figure 3: Trajectories near $(0, 0)$ of system (21). Upper left: $\beta = -1.25$, spiral source. Upper right: $\beta = -0.25$, spiral sink. Lower left: $\beta = 0.25$, saddle. Lower right: $\beta = 3.5$, source.