

Math 312 Lecture Notes

One Dimensional Maps

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An Example

We begin with the simplest model of population growth. Suppose, for example, a population increases by 15 percent each year. Let p_n be the population at the end of year n , and assume that p_0 is given. Then an increase of 15 percent each year gives

$$p_{n+1} = 1.15p_n. \quad (1)$$

If $p_0 = 100$, then $p_1 = 1.15(100) = 115$, $p_2 = 1.15(115) = 132.3$, $p_3 = 1.15(132.3) = 152.1$, and so on. Table 1 shows the first nine iterations of this process. The data is plotted in Figure 1.

n	p_n	n	p_n
0	100.0	5	201.1
1	115.0	6	231.3
2	132.3	7	266.0
3	152.1	8	305.9
4	174.9	9	351.8

Table 1: Iteration of the map (1), with $p_0 = 100$.

We derive a formula for p_n by noting that

$$\begin{aligned} p_1 &= 1.15p_0 \\ p_2 &= 1.15p_1 = (1.15)^2 p_0 \\ p_3 &= 1.15p_2 = (1.15)^3 p_0 \\ &\vdots \\ p_n &= (1.15)^n p_0 \end{aligned} \quad (2)$$

Thus we have the familiar “exponential growth” of the population.

More generally, the same argument shows that the solution to

$$p_{n+1} = ap_n, \quad (3)$$

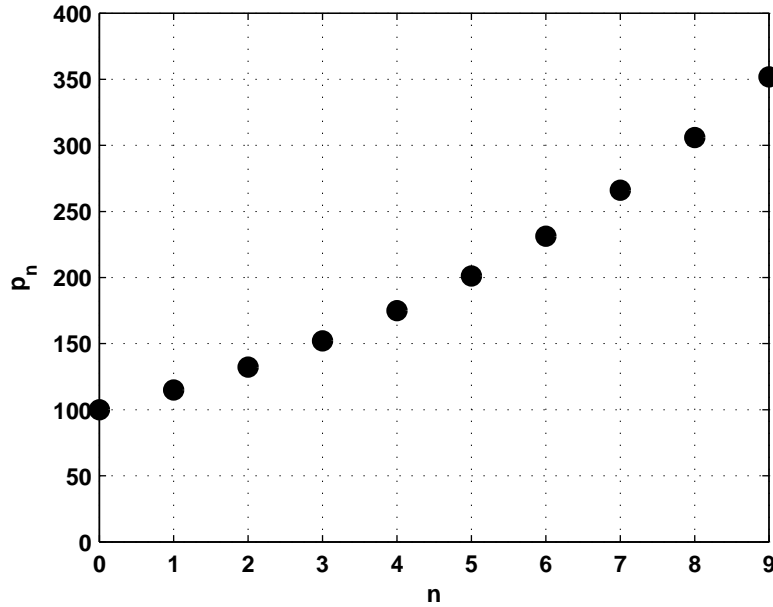


Figure 1: Plot of the data for the simple population growth model (1). The data is in Table 1.

given p_0 , is

$$p_n = a^n p_0. \quad (4)$$

Let's compare the discrete time map to the continuous time model. Recall that a first order differential equation that leads to exponential growth (or decay) is

$$\frac{dp}{dt} = rp, \quad p(0) = p_0, \quad (5)$$

which has the solution

$$p(t) = p_0 e^{rt} = (e^r)^t p_0. \quad (6)$$

I've written the solution as $(e^r)^t p_0$ to make clear the analogy between the continuous solution and the solution to the discrete model in (4). The term e^r is analogous to a , and t is analogous to n .

Let's rewrite (1) as

$$p_{n+1} = (1 + 0.15)p_n = p_n + 0.15p_n \quad (7)$$

or

$$p_{n+1} - p_n = 0.15p_n. \quad (8)$$

This form of the equation expresses the *discrete change* in p_n as a function of p_n . (Such an equation is often called a *difference equation*.) Compare this to equation (5), which gives the *instantaneous rate of change* of p as a function of p .

One-dimensional Maps

The general one-dimensional map has the form

$$x_{n+1} = f(x_n), \quad \text{given } x_0. \quad (9)$$

n	x_n	n	x_n
0	0.1	5	0.6771
1	0.2520	6	0.6122
2	0.5278	7	0.6648
3	0.6978	8	0.6240
4	0.5904	9	0.6570

Table 2: Iteration of the logistic map (10), with $r = 2.8$ and $x_0 = 0.1$.

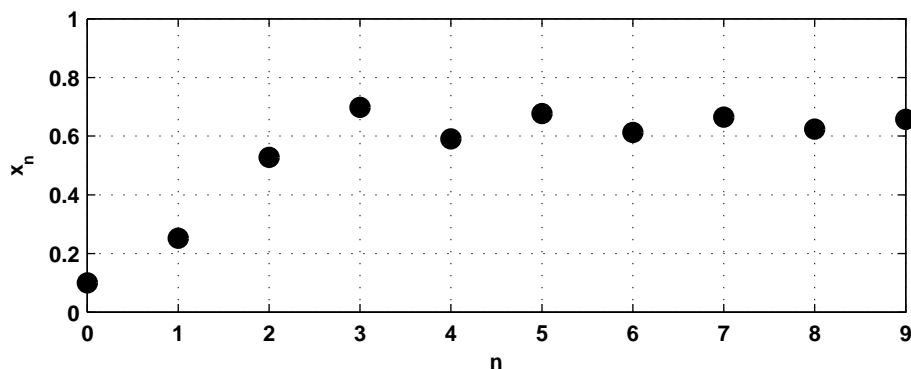


Figure 2: The first several iterates of the logistic map (10) with $r = 2.8$ and $x_0 = 0.1$. This is a plot of the data shown in Table 2.

We'll develop several tools for studying such maps.

“Cobwebbing”: The Graphical Iteration Procedure

There is a simple graphical procedure for generating the iterations of a one-dimensional map. As an example, we consider the logistic map

$$x_{n+1} = rx_n(1 - x_n) \quad (10)$$

where $r > 0$ is a constant. Table 2 shows the first few iterates in the sequence that results when $r = 2.8$ and $x_0 = 0.1$. Figure 2 shows the plot of x_n as a function of n .

The graphical technique for finding the iterations of this map begins by plotting the graph of the $f(x) = rx(1 - x)$, as in Figure 3 (where $r = 2.8$). Given x_0 , we draw a line from the $x = x_0$ on the x axis up the the graph of x_0 to obtain x_1 . To obtain x_2 , we need to go to $x = x_1$ on the x axis. This can be done by drawing a line horizontally from (x_0, x_1) to the line $y = x$. See the upper left plot of Figure 4. To find x_2 , we again draw a line vertically at $x = x_1$ to the graph, and from the graph we draw a line horizontally to $y = x$. See the upper right plot of Figure 4. We continue this process to generate further iterations, as in the lower left and right plots of Figure 4. In these plots, I've included vertical lines from the x axis up to the graph, but we don't really need these lines. In practice, the procedure is: draw a line vertically to the graph, then draw a line horizontally to $y = x$, and then repeat the process. An example is shown in Figure 5.

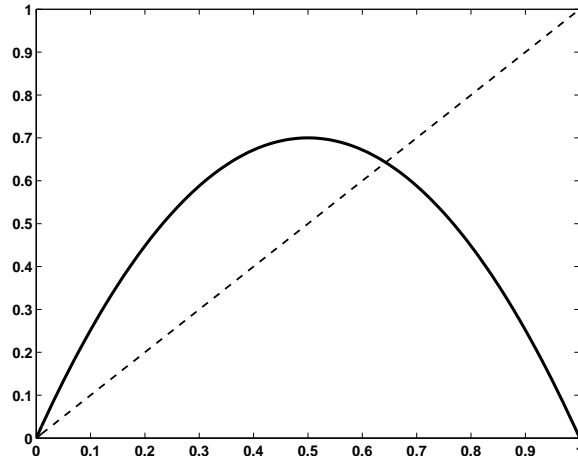


Figure 3: A graph of the logistic map $rx(1-x)$ for $r = 2.8$.

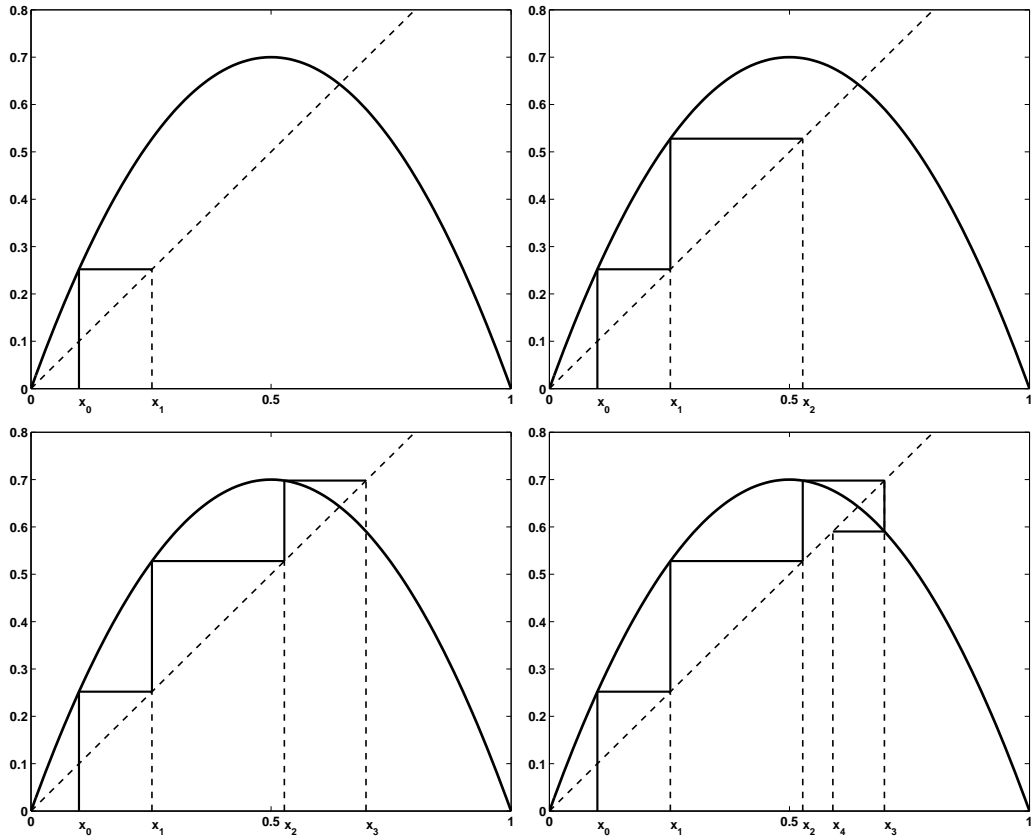


Figure 4: A sequence of plots that illustrate “cobwebbing”. The initial point is $x_0 = 0.1$.

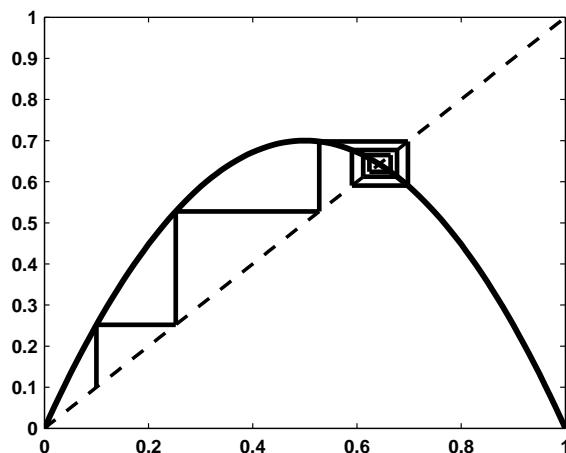


Figure 5: When cobwebbing, we usually don't bother dropping lines down to the horizontal axis. After the first vertical line is drawn from (x_0, x_0) to the graph, we draw a line horizontally to the line $y = x$, and then another line vertically to graph, and repeat. This cobweb plot shows the same data listed in Table 2 and plotted in Figure 2.

Linear Maps

A *linear* one-dimensional map is

$$x_{n+1} = ax_n \quad (11)$$

where a is a constant. (The population model (1) is an example.) The solution is

$$x_n = a^n x_0. \quad (12)$$

Let's look at how the behavior of the solution depends on a .

1. If $a > 1$, then $a^n > 0$, and a^n increases without bound as n increases. So x_n grows exponentially.
2. If $a = 1$, then $a^n = 1$ for all n , so $x_n = x_0$.
3. If $0 < a < 1$, then $a^n > 0$, and a^n approaches zero as n increases; x_n decays to zero monotonically.
4. If $a = 0$, then $x_n = 0$ for all $n > 0$.
5. If $-1 < a < 0$, then a^n alternates sign, and a^n approaches zero as n increases; x_n decays to zero while alternating sign.
6. If $a = -1$, then $a^n = (-1)^n$, which is 1 if n is even and -1 if n is odd. Thus x_n alternates between x_0 and $-x_0$.
7. If $a < -1$, then a^n alternates sign, and $|a^n|$ increases without bound as n increases. So x_n alternates sign while $|x_n|$ grows exponentially.

Overall, the magnitude of x_n *grows* if $|a| > 1$, and *decays* if $|a| < 1$. Figures 6 and 7 show sample plots of x_n and cobweb diagrams for the various cases.

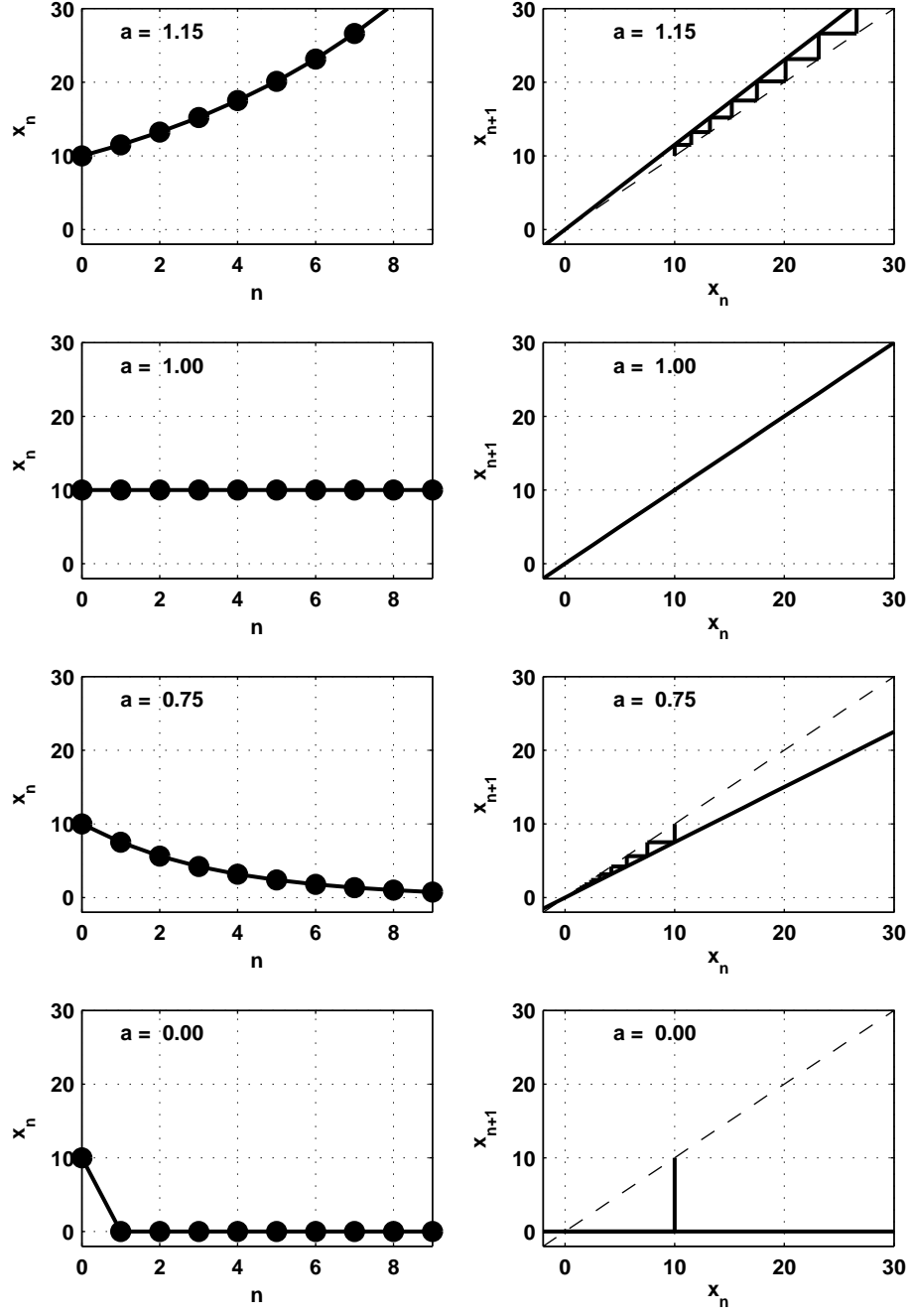


Figure 6: Examples of linear maps, $a \geq 0$. In all cases, $x_0 = 10$.

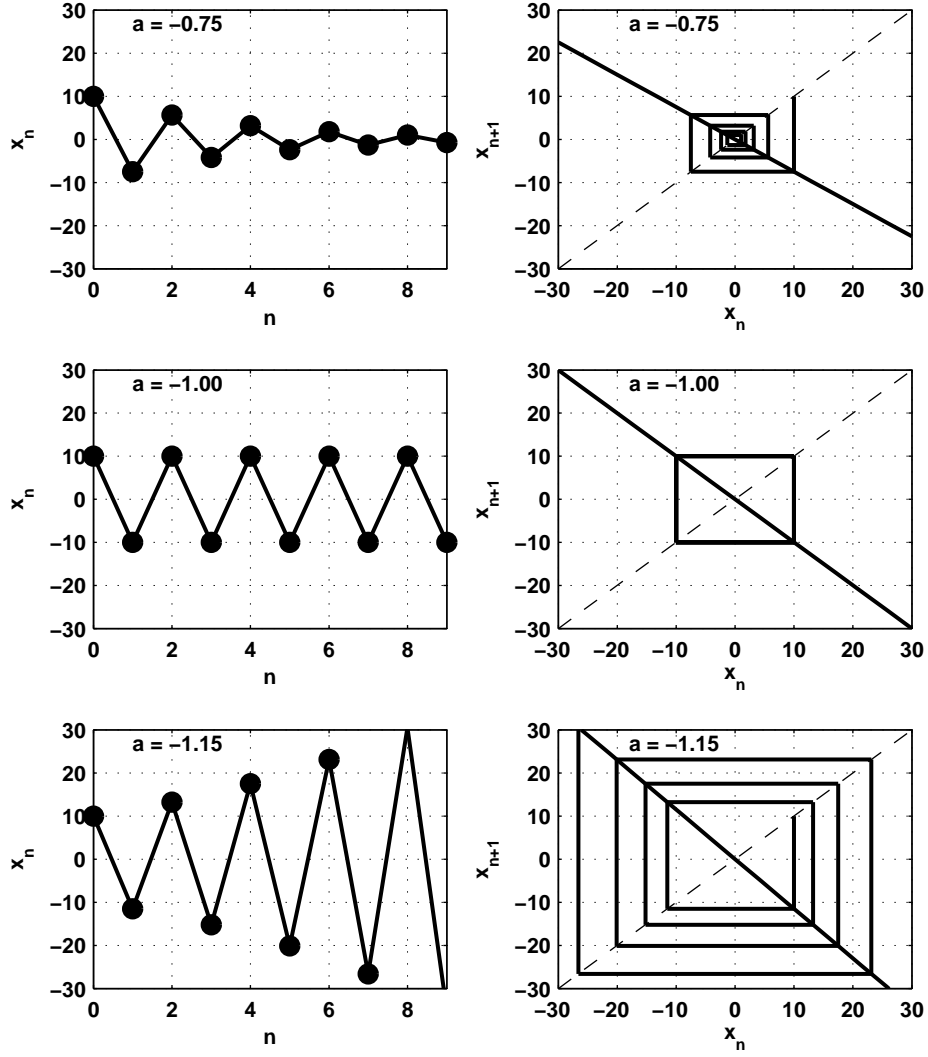


Figure 7: Examples of linear maps, $a < 0$. In all cases, $x_0 = 10$.

Fixed Points, Stability, Linearization

A *fixed point* of the map (9) is a point x^* where $f(x^*) = x^*$. If $x_0 = x^*$, then $x_n = x^*$ for all $n > 0$, so the fixed points of f are the constant solutions to (9).

In the logistic map shown in Figure 3, we see there are two points where $f(x) = x$. These are the points where the graph of f crosses the line $y = x$. For the example shown in Figure 3, the graph is $f(x) = 2.8x(1 - x)$, so to find the fixed points, we must solve

$$2.8x(1 - x) = x \implies -2.8x^2 + 1.8x = 0 \implies x = 0 \text{ or } x = \frac{1.8}{2.8} \quad (13)$$

So the two fixed points are $x = 0$ and $x = \frac{1.8}{2.8} \approx 0.643$.

Definition. The fixed point x^* is a *sink* or *attractor* if there is a neighborhood N of x^* such that $x_n \rightarrow x^*$ for all x_0 in N . We also say x^* is *asymptotically stable*.

Definition. The fixed point x^* is a *source* or *repellor* if there is a neighborhood N of x^* such that if x_0 is in N , then x_n eventually leaves N .

Definition. The fixed point x^* is *unstable* if for every neighborhood N of x^* , there are points arbitrarily close to x^* whose iterates leave N .

Note that a source is unstable, but an unstable fixed point is not necessarily a source. See, for example, Figure 10.

Behavior near a fixed point: the linearization. Let x^* be a fixed point of (9). If x is close to x^* , we can approximate $f(x)$ with the tangent line at x^* :

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*). \quad (14)$$

Let $u_n = x_n - x^*$ (i.e. $x_n = x^* + u_n$). Then, by replacing x_n with $x^* + u_n$ in (9), we obtain

$$x^* + u_{n+1} = f(x^* + u_n) \approx f(x^*) + f'(x^*)u_n. \quad (15)$$

Since $f(x^*) = x^*$, we can cancel x^* on the left and $f(x^*)$ on the right. We are left with the *linearization of the map at x^** :

$$u_{n+1} = f'(x^*)u_n \quad (16)$$

This is a linear map. We have already seen how the solution to the linear map depends on $a = f'(x^*)$; see Figures 6 and 7. However, the linearization is just an approximation to the actual map (9). The following theorem tells us when the linear approximation is “good enough” to classify the stability of the fixed point x^* of (9).

Theorem 1. (i) If $|f'(x^*)| < 1$, then x^* is a sink. (ii) If $|f'(x^*)| > 1$, then x^* is a source.

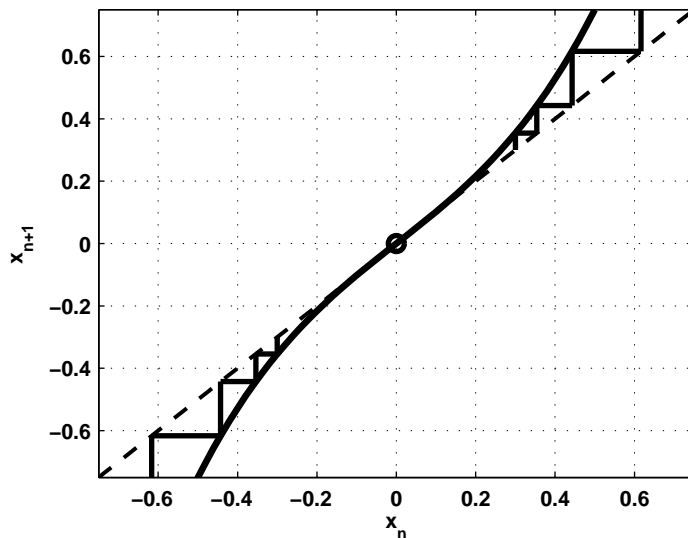


Figure 8: An example that shows why we can not determine the stability of a fixed point based on the linearization when $|f'(x^*)| = 1$. The fixed point is $x^* = 0$, and $f'(x^*) = 1$. In this case, x^* is a source.

Example. Consider again the logistic map (10) with $r = 2.8$. The graph is shown in Figure 3, and earlier we found the fixed points to be $x_1^* = 0$ and $x_2^* = \frac{1.8}{2.8}$. We find

$$f'(x) = 2.8(1 - 2x). \quad (17)$$

We use Theorem 1 to determine the stability of the fixed points.

- At x_1^* , we have $f'(0) = 2.8 > 1$, so by Theorem 1, x_1^* is a source.
 - At x_2^* , we have $f'(1.8/2.8) = -0.8$, so by Theorem 1, x_2^* is a sink.
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If $|f'(x^*)| = 1$, we can not make any conclusions about the stability of the fixed point. Examples demonstrating why this is the case are shown in Figures 8, 9 and 10.

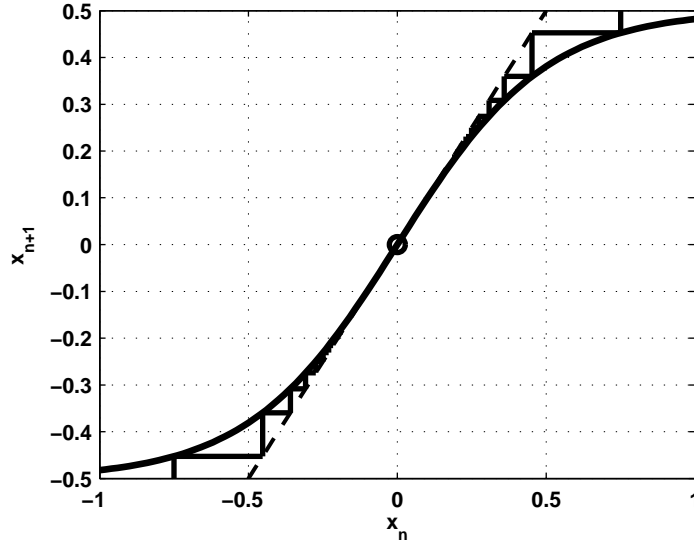


Figure 9: An example that shows why we can not determine the stability of a fixed point based on the linearization when $|f'(x^*)| = 1$. The fixed point is $x^* = 0$, and $f'(x^*) = 1$. In this case, x^* is a sink.

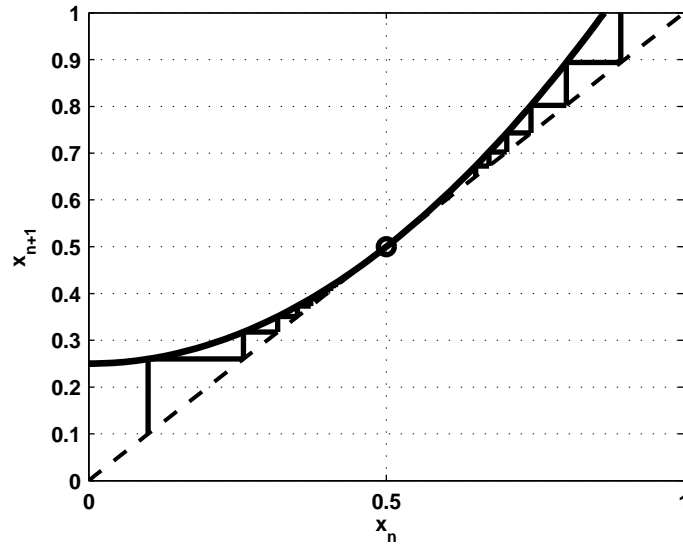


Figure 10: Another example that shows why we can not determine the stability of a fixed point based on the linearization when $|f'(x^*)| = 1$. In this case, $x^* = 1/2$ is a fixed point, and $f'(x^*) = 1$. The iterations of points close to but less than x^* converge to x^* , but iterations of points that are greater than x^* diverge from x^* . In this case, x^* is unstable (but it is not a source).